A New Bayesian Time-Predictable Modeling of Eruption Occurrences and Forecasting: Application to Mt Etna volcano and Kilauea volcano

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Abstract

In this paper we propose a model to forecast eruptions in a real forward perspective. Specifically, the model provides a forecast of the next eruption after the end of the last one, using only the data available up to that time. We focus our attention on volcanoes with open conduit regime and high eruption frequency. We assume a generalization of the classical time predictable model to describe the eruptive behavior of open conduit volcanoes and we use a Bayesian hierarchical model to make probabilistic forecasts. We apply the model to Kilauea volcano eruptive data and Mount Etna volcano flank eruption data.

The aims of the proposed model are: 1) to test whether or not the Kilauea and Mount Etna volcanoes follow a time predictable behavior; 2) to discuss the volcanological implications of the time predictable model parameters inferred; 3) to compare the forecast capabilities of this model with other models present in literature. The results obtained using the MCMC sampling algorithm show that both volcanoes follow a time pre-
dictable behavior. The numerical values inferred for the parameters of the
time predictable model suggest that the amount of the erupted volume
could change the dynamics of the magma chamber refilling process during
the repose period. The probability gain of this model compared with other
models already present in literature is appreciably greater than zero. This
means that our model provides better forecast than previous models and it
could be used in a probabilistic volcanic hazard assessment scheme.

Keywords: Effusive volcanism, Bayesian hierarchical modeling, Mount
Etna, Kilauea, Probabilistic forecasting, Volcanic hazards.

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1. Introduction

One of the main goals in modern volcanology is to provide reliable forecast of volcanic eruptions with the aim of mitigating the associated risk. The extreme complexity and non linearity of a volcanic system make deterministic prediction of the evolution of volcanic processes rather impossible (e.g. Marzocchi 1996; Sparks 2003). Volcanic systems are intrinsically stochastic. In general, eruption forecasting involves two different time scales: i) a short-term forecasting, mostly based on monitoring measures observed during an episode of unrest (e.g., Newhall and Hoblitt 2002, Marzocchi et al. 2008 among others); ii) a long-term forecasting, usually made during a quiet period of the volcano, and mostly related to a statistical description of the past eruptive catalogs (e.g. Klein, 1982, Bebbington and Lai, 1996a among others). Here, we focus our attention only on this second issue.

In a long-term eruption forecast perspective we believe that an incisive and useful forecast should be made before the onset of a volcanic eruption, using the data available at that time, with the aim of mitigating the associated volcanic risk. In other words, models implemented with forecast purposes have to allow for the possibility of providing “forward” forecasts and should avoid the idea of a merely “retrospective” fit of the data available. Models for forecasting eruptions should cover a twofold scope: fit the eruption data and incorporate a robust forecast procedure. While the first requirement is mandatory, the latter one is not commonly used in statistical modeling of volcanic eruptions. By carrying out and testing a forecast procedure on data available at the present, one could make enhancement in the forecast matter and reveal the model limitations.

Different methods have been presented in the past years aiming at
the identification of possible recurrence or correlation in the volcanic time
and/or volume data for long-term eruption forecast. Klein (1982), Mulargia
et al (1985) and Bebbington and Lai (1996a and 1996b) studied the time
series of volcanic events looking at the mean rate of occurrence. Sandri et al.
(2005) applied a generalized form of time predictable model to Mount Etna
eruptions using regression analysis. Marzocchi and Zaccarelli (2006) found
different behavior for volcanoes with “open” conduit regime compared to
those with “closed” conduit regime. Open conduit volcanoes ( Mt Etna,
Kilauea volcano there tested) seem to follow a so-called Time Predictable
Model. While closed conduit volcanoes seem to follow a homogeneous Pois-
son process. De La Cruz-Reyna (1991) proposed a load-and-discharge model
for eruptions in which the time predictable model could be seen as a partic-
ular case. Bebbington (2008) presented a stochastic version of the general
load-and-discharge model also including a way to take into account the his-
tory of the volcano discharging behavior. In this paper the author studied
the time predictability as a particular case of his model with application to
Mount Etna, Mauna Loa and Kilauea data series. A different hierarchical
approach has been presented by Bebbington (2007) using Hidden Markov
Model to study eruption occurrences with application to Mount Etna flank
eruptions. This model is able to find any possible underlying volcano ac-
tivity resulting in changes of the volcanic regime. Salvi et al (2006) carried
out analysis for Mt Etna flank eruption using an Non Homogeneous Poisson
process with a power law intensity, while Smethurst et al (2009) applied a
Non Homogeneous Poisson process with a piecewise linear intensity to Mt
Etna flank eruptions.

In a recent paper Passarelli et al (2010) proposed a Bayesian Hierarchical
Model for interevent time-volumes distribution using the time predictable process with application to Kilauea volcano. The model presents a new Bayesian methodology for an open conduit volcano that accounts for uncertainties in observed data. Besides, the authors present and test the forecast ability of the model retrospectively on the data through a forward sequential procedure. While the model seems to produce better forecasts that other models in the literature, it produces fits to eruption volumes and interevent times that are too large, reducing the forecast performances. This is due to the use of normal distributions for the log-transformed data. This is a restrictive distributional assumption that creates very long tails. Here we propose a more general modeling strategy that allows for more flexible distributions for the interevent times and volumes data.

Using the same framework of Passarelli et al (2010), we will model the interevent times and volumes data through distributions with exponential decay (Klein, 1982, Mulargia, 1985, Marzocchi, 1996, Bebbington, 1996a, 1996b and 2007, Salvi et al, 2006, Smethurst et al, 2009). This provide a general treatment of the volume and interevent time series, hopefully improving the forecast capability of the model. As eruptive behavior we use the Generalized Time Predictable Model (Sandri et al, 2005 and Marzocchi and Zaccarelli, 2006). This model assumes: 1) eruptions occur when the volume of magma in the storage system reaches a threshold value, 2) magma recharging rate of the shallow magma reservoir could be variable and 3) the size of eruptions is a random variable, following some kind of statistical distribution. Under these assumptions, the time to the next eruption is determined by the time required for the magma entering the storage system to reach the eruptive threshold. The more general form for a time-predictable
model is a power law between the erupted volume and the interevent time:

\[ r_i = c e^{b_i} \quad (1) \]

where, if the parameter \( b \) is equal to unity we are in a classical time predictable system (see De La Cruz Reyna 1991, Burt et al. 1994). If \( b \) is equal to 0 the system is not time predictable. If \( b > 1 \) or \( 0 < b < 1 \) we have a non-linear relationship implying a longer or shorter interevent time after a large volume eruption compared to a classical time predictable system. The goal of the present work is to infer the parameters of Equation (1).

In the remainder of this paper, we focus our attention on some specific issues: 1) to discuss the physical meaning and implications of parameters inferred; 2) to verify if the model describes the data satisfactorily; 3) to compare the forecasting capability of the present model with other models previously published in literature using the sequential forward procedure discussed in Passarelli et al. (2010). In the first part of this paper, we will introduce the generality of the model by considering three stages: 1) a model for the observed data; 2) a model for the process and 3) a model for the parameters (Wikle, 2002). Then we will discuss how: 1) to simulate the variables and parameters of the model; 2) to check the model fit; 3) to use the model to assess probabilistic forecast in comparison with other statistical published models. The last part of the paper contains the application of the model to Kilauea volcano and Mount Etna eruptive data.
2. A Bayesian Hierarchical Model for Time-Predictability

In the following sections we present a detailed description of our proposed model. We denote it as Bayesian Hierarchical Time Predictable Model II (BH_TPM II), while the model proposed in Passarelli et al. (2010) is denoted as BH_TPM. In Section 2.1 we discuss the measurement error model. In Section 2.2 we consider a model for the underlying process, which is based on the exponential distribution. In Section 2.3 we discuss the distributions that are placed on the parameters that control the previous two stages of the model. In Section 2.4 we introduce the simulation procedure and in Section 2.5 we consider model assessment and forecasting of volcanic eruptions.

2.1. Data model

The dataset for this model has \( n \) pairs of observations: volumes and interevent times denoted as \( d_v \) and \( d_r \), respectively. We assume independence between the measurement errors of interevent times and volumes. This is justified by the fact that these two quantities are measured using separate procedures. Dependence between times and volumes will be handled at the process stage, following the power law in Equation (1). In addition, we assume that, conditional on the process parameters, the interevent times or volumes are independent within their group. This is a natural assumption within a hierarchical model framework. It is equivalent to assuming that the volumes (times) are exchangeable between them. Exchangeability implies that all permutations of the array of volumes (times) will have the same joint distribution. Exchangeability is weaker than independence, and it is
Our measurement error model assumes a multiplicative error for the observations. This follows from BH_TPM where it was assumed that

$$\log(d_{ri}) = \log r_i + \log \epsilon_{ri}$$  \hspace{1cm} (2)$$

with \(\log \epsilon_{ri} \sim N(0, \sigma^2_{Dri})\) where \(\sigma^2_{Dri} = (\frac{\Delta d_{ri}}{d_{ri}})^2\) (for more details see Pas-sarelli et al, 2010). The analogous assumption \(\log(d_{vi}) = \log v_i + \log \epsilon_{vi}\) and \(\log \epsilon_{vi} \sim N(0, \sigma^2_{Dvi})\) where \(\sigma^2_{Dvi} = (\frac{\Delta d_{vi}}{d_{vi}})^2\), was considered for the volumes. Exponentiating on both sides of Equation (2) we have

$$d_{ri} = \epsilon_{ri}r_i$$  \hspace{1cm} (3)$$

which is the data stage model we propose in BH_TPM II.

The error in Equation (3) follows a probability distribution with positive support. We choose an inverse gamma distribution. This is a flexible distribution defined by two parameters which will provide computational advantages. We fix the two defining parameters by assuming \(E(\epsilon_{ri}) = 1\) and calculating \(\text{var}(\epsilon_{ri})\) using a delta method approximation. Specifically, from the assumption that \(\log \epsilon_{ri} \sim N(0, \sigma^2_{Dri})\), we have that \(E(\log \epsilon_{ri}) = 0\) and \(\text{var}(\log \epsilon_{ri}) = \sigma^2_{Dri} = (\frac{\Delta d_{ri}}{d_{ri}})^2\). Thus

$$\text{var}(\epsilon_{ri}) = \sigma^2_{Dri} \left[ g'(\text{E} \left( \frac{\Delta d_{ri}}{d_{ri}} \right)) \right]^2 = \left( \frac{\Delta d_{ri}}{d_{ri}} \right)^2$$

where \(g(x) = \exp(x)\) and \(g'\) is the first derivative.

Recall that a random variable \(X\) that follows an inverse gamma distribution with parameters \(\alpha_{ri}\) and \(\beta_{ri}\) has expected value \(E(X) = \frac{\beta_{ri}}{\alpha_{ri}-1}\) and
variance \( \text{var}(X) = \frac{\beta_r^2}{(\alpha_r - 1)^2(\alpha_r - 2)} \). We then have that
\[
\frac{\beta_r}{\alpha_r - 1} = 1 \quad \text{and} \quad \frac{\beta_r^2}{(\alpha_r - 1)^2(\alpha_r - 2)} = \left( \frac{\Delta d_{r_i}}{d_{r_i}} \right)^2.
\]
Solving for \( \alpha_r \) and \( \beta_r \) gives
\[\alpha_r = \left( \frac{d_{r_i}}{\Delta d_{r_i}} \right)^2 + 2 \quad \text{and} \quad \beta_r = \left( \frac{d_{r_i}}{\Delta d_{r_i}} \right)^2 + 1 \] where \( \Delta d_{r_i} / d_{r_i} \) is the relative error. Analogous calculations can be done for the volumes.

The joint distributions for the measurement errors \( \epsilon_r = (\epsilon_{r_1}, \ldots, \epsilon_{r_n}) \) and
\[\epsilon_v = (\epsilon_{v_1}, \ldots, \epsilon_{v_n}) \]
result in
\[
[\epsilon_r|\alpha_r, \beta_r] = \prod_{i=1}^n \Gamma^{-1}(\alpha_r, \beta_r) \quad \text{and} \quad [\epsilon_v|\alpha_v, \beta_v] = \prod_{i=1}^n \Gamma^{-1}(\alpha_v, \beta_v) \quad (4)
\] where \( \alpha_{v_i} = \left( \frac{d_{v_i}}{\Delta d_{v_i}} \right)^2 + 2 \) and \( \beta_{v_i} = \left( \frac{d_{v_i}}{\Delta d_{v_i}} \right)^2 + 1 \). Here we use \([X]\) to denote the distribution of a random variable \( X \) and \( \Gamma^{-1} \) to denote an inverse gamma.

The distribution of the observed variables \( d_{r_i} \) and \( d_{v_i} \) can be obtained from the error distributions specified by the expression in (4). Noting that
\[
\left| \frac{d_{r_i}}{d(d_{r_i})} \right| = \frac{1}{r_i}
\] and using the change of variables formula for probability density functions, we have that
\[
[d_r|\alpha_r, \beta_r, r_i] = \prod_{i=1}^n \Gamma^{-1}(\alpha_r, \beta_r, r_i) \quad \text{and} \quad [d_v|\alpha_v, \beta_v, v_i] = \prod_{i=1}^n \Gamma^{-1}(\alpha_v, \beta_v, v_i). \quad (5)
\] The expression in (5) will be used to obtain the likelihood function for our data.

2.2. Process model

The starting point for the model pertaining the unobserved quantities \( r_i \) is the assumption that volcanic eruptions correspond to a homogeneous Poisson process. A Poisson process in time has the property that the
number of events that occur during a given time interval follow a Poisson
distribution with mean proportional to the length of the interval. Additionally
the time between consecutive events is distributed as an exponential
random variable (Klein, 1982, Mulargia, 1985, Marzocchi, 1996, Bebbington
and Lai, 1996a, 1996b). Thus we assume that \( r_i \sim \text{Exp}(\lambda) \) implying that
the joint distribution of \( r = (r_1, \ldots, r_n) \) is given by \( [r|\lambda] = \prod_{i=1}^{n} \text{Exp}(\lambda) \).
Given the distributional assumption for the interevent times we can obtain
the distribution of the volumes \( v_i \) using Equation (1). Recalling that
\( r_i = cv_i^b \) and \( \left| \frac{dv_i}{dr_i} \right| = cbv_i^{b-1} \), the change of variable formula for probability
density functions yields \( [v_i] = cb\lambda v_i^{b-1}e^{-\lambda cv_i^b} \). Written in distributional form
we have: \( v_i \sim \text{Wb} \left( b, \left( \frac{1}{\lambda c} \right)^{\frac{1}{b}} \right) \) where \( \text{Wb}(\cdot, \cdot) \) denotes a Weibull distribution.
The joint distribution for the volumes \( v = (v_1, \ldots, v_n) \) is given as
\[ [v|\lambda, b, c] = \prod_{i=1}^{n} \text{Wb} \left( b, \left( \frac{1}{\lambda c} \right)^{\frac{1}{b}} \right) \].
This completes the specification of the second stage of our model.

2.3. Parameters model

To complete our model we need to specify distributions for the parameters \( b, c \) and \( \lambda \). Our choices are based on prior information obtained from
previous modeling efforts. In a Bayesian setting, like the one proposed in
this work, we have the ability to include structural information, like the
one used to build the second stage model, as well as prior information. The
final product consists of the posterior distribution of all model parameters.
This contains a blend of the information provided by all the stages of the
model: data, process and prior knowledge.

We choose for \( \lambda \) a gamma distribution with known parameters, from
now on hyperparameters. This is denoted as have: \( \lambda \sim \text{Ga}(\alpha_\lambda, \beta_\lambda) \) where
\( \alpha_\lambda \) and \( \beta_\lambda \) are calculated by fitting the interevent times data with a gamma distribution, via maximum likelihood estimation. For the time predictable equation parameters, i.e. \( b \) and \( c \), we use normal distributions with moments calculated using the posterior distributions taken from BH_TPM (Passarelli et al., 2010). Thus \( [b] = N(\mu_b, \sigma^2_b) \) and \( [c] = N(\mu_c, \sigma^2_c) \).

By choosing the values of the hyperparameters we are introducing a certain degree of subjectivity in our modeling. We believe that this is a desirable feature of the Bayesian approach, as it allows to incorporate knowledge from similarly behaved open conduit volcanoes. We remark the subjective approach allowed in Bayesian Statistics could be a suitable tool in modeling geophysical phenomena where available data are scarce. This provides the possibility of incorporating knowledge obtained from other sources in a probabilistic way, through the prior distributions. This allows for the introduction of physical and/or statistical constraints, when available, on the parameters governing the examined phenomenon. In principle this methodology could be helpful to improve the understanding of a particular system. We want to point out, though, that subjective statistical modeling choices need careful justification, possibly relying on physical or phenomenological constraints.

2.4. Posterior and full conditional distributions

The three stage model specification developed in the previous sections produces a posterior distribution for the model parameters \( r, v, b, c \) and \( \lambda \) that, using Bayes theorem, can be written as

\[
[r, v, b, c, \lambda \mid d_r, d_v, \Delta d_r, \Delta d_v] \propto \]

\[
[d_r \mid \alpha_{d_r}, \beta_{d_r}, r][d_v \mid \alpha_{d_v}, \beta_{d_v}, v][v \mid c, \lambda, b][r \mid \lambda][\lambda][b][c] .
\]
To make inference about the posterior distribution specified by Equation (7) we draw samples from it using Markov chain Monte Carlo (MCMC) methods (Gelman et al. 2000, Gilks et al. 1996). This requires the full conditional distributions for each parameter in the model. In the equations below we specify each of them using the notation \([X|\ldots]\) to indicate the distribution of variable \(X\) conditional on all other variables.

\[
[r_i|\ldots] \propto r_i^{\alpha_{r_i}} \exp \left\{ -r_i \left( \lambda + \frac{\beta_{r_i}}{d_{r_i}} \right) \right\} = \text{Ga} \left( \alpha_r + 1, \lambda + \frac{\beta_{r_i}}{d_{r_i}} \right)
\]

\[
[v_i|\ldots] \propto v_i^{\alpha_{v_i} + b} \exp \left\{ \left( \lambda c v_i^b + \frac{\beta_{v_i} v_i}{d_{v_i}} \right) \right\}
\]

\[
[\lambda|\ldots] \propto \lambda^{2n + \alpha_{\lambda} - 1} \exp \left\{ -\lambda \left( \beta_{\lambda} + c \sum_{i=1}^{n} v_i^b + \sum_{i=1}^{n} r_i \right) \right\} = \text{Ga} \left( \alpha_{\lambda} + 2n, \beta_{\lambda} + c \sum_{i=1}^{n} v_i^b + \sum_{i=1}^{n} r_i \right)
\]

\[
[c|\ldots] \propto c^n \exp \left\{ -c \lambda \sum_{i=1}^{n} v_i^b + \frac{\mu_c c}{2\sigma_c^2} - \frac{c^2}{2\sigma_c^2} \right\}
\]

\[
[b|\ldots] \propto \prod_{i=1}^{n} (b v_i^{b-1}) \exp \left\{ -\lambda c \sum_{i=1}^{n} v_i^b + \frac{\mu_b b}{2\sigma_b^2} - \frac{b^2}{\sigma_b^2} \right\}
\]

The full conditional distributions of \(r_i, i = 1, \ldots, n\) and \(\lambda\) can be sampled directly in Gibbs steps, as they correspond to gamma distributions. The full conditionals of the other parameter do not have standard forms. So
we use Metropolis steps to obtain samples from them. Once samples from
the MCMC are obtained we discard the first part of the chain as a burn-in
phase (see for example Gilks et al., 1996); then we do a “thinning” of the
chain by subsampling the simulated values at a fixed lag $k$. This strategy
ensures that, setting $k$ to some high enough value, successive draws of the
parameters are approximately independent (Gelman, 1996). To define the
lag we use the auto-correlation function as shown later in the text.

2.5. Model Checking and Forecasting procedure

We have presented, so far, the hierarchical structure of the model and
the fitting procedure for the model parameters, based on MCMC sampling.
We now address the issues of (1) testing the goodness of the proposed model
and (2) forecasting future interevent times.

Bayesian model checking is based on the idea that predictions obtained
from the model should be compatible with actual data. So our strategy
consists of simulating data from the predictive posterior distribution and
comparing them to actual observations. The predictive posterior distribu-
tion quantifies the uncertainty in future observations given the observed
data. By denoting $\hat{r}$ future values of interevent times we have that the
posterior predictive is

$$
[\hat{r} | \text{Data}] = \int_{\mathbb{R}^+} [\hat{r} | \lambda][\lambda | \text{Data}] d\lambda
$$

where $\mathbb{R}^+$ denotes the parameter space. To obtain samples from the distri-
bution in Equation (8) we start from the MCMC samples of $\lambda$. Suppose we
have $N$ of them and denote them as $\lambda^j$. Conditional on $\lambda^j$, for $j = 1, \ldots, N$
we simulate $\hat{r}^j$ from $[\hat{r} | \lambda^j]$, which are products of exponentials. We ob-
tain $N$ synthetic catalogs with $n$ pairs of interevent time and volume data.
These are compared to the observed data using descriptive statistics. As
descriptive statistics we choose the mean number of events or rate of occurre-
rence, maximum, minimum, median and standard deviation for both real
and synthetic data.

To test the ability of the model to forecast future volumes and interevent
times we use a sequential approach. We proceed by fitting the model to the
first data pair, then we add the data of the second event to the model
fitting. We continue adding data sequentially until the last event. This
provides an assessment of the number of data needed for the model to effec-
tively “learn” the model parameters. Therefore, we are able to decide the
minimum amount of data needed to define the learning phase for the model.

For the remaining part of data (i.e. voting phase), we use the sequentially
sampled parameters to generate the distribution for the next event (in-
terevent time). We can thus compare the forecasted interevent times with
the observed data and with forecasts from other published methods (see
forward procedure discussed in Passarelli et al, 2010).

A close look at Equation (8) reveals a practical forecasting problem. We
observe that the posterior predictive distribution of the interevent times
depends on the distribution of the interevent times given the parameter
\( \lambda \). While this is statistically correct, it is not a realistic forecasting pro-
cedure. In fact, in a generalized time predictable system the time to the
next eruption is strongly dependent on the volume of the previous eruption.
More explicitly, in our current framework, after the end of the \( n \)-th eruption
we have samples of \( \lambda \) that are simulated using only the information up to

\((d_{r(n-1)}, d_{v(n-1)})\). We would like to incorporate the information on \( d_{v_n} \). We
do this by resampling the posterior realizations of \( \lambda \) using the Sampling
Importance Resampling algorithm (hereafter SIR), (Rubin, 1988, Smith and Gelfand, 1992) together with Bayes theorem.

Let $\theta_{n-1} = b, c, \lambda$ be the samples obtained from our model using the first $n-1$ data. For the $n$-the interevent time we have

$$[\tilde{r}_n \mid d_v] = \int_{R^+} [\tilde{r}_n \mid d_v, v_{n-1}, \theta_{n-1}] [\theta_{n-1} \mid d_v, v_{n-1}] d\theta_{n-1} \quad (9)$$

Obtaining samples from the predictive distribution in Equation (9) requires samples of $[\theta_{n-1} \mid d_v, v_{n-1}]$, which are not available. Our MCMC algorithm produces samples of $[\theta_{n-1} \mid d_{v_{n-1}}, v_{n-1}]$ instead. Using Bayes theorem we have that

$$[\theta_{n-1} \mid d_v, v_{n-1}] \propto [d_v \mid v_{n-1}, \theta_{n-1}] [\theta_{n-1} \mid v_{n-1}] \quad (10)$$

In Equation (10) we recognize $[d_v \mid v_{n-1}, \theta_{n-1}]$ as the inverse gamma distribution used for volume data in Equation (5). $[\theta_{n-1} \mid v_{n-1}]$ is the posterior distribution for parameters $\lambda, b$ and $c$ up to the first $n-1$ events. The SIR algorithm consists of resampling the output from the MCMC, say $\theta^j_{n-1}$, with replacement, using the normalized weights defined as

$$w^*(\theta^j_{n-1}) = \frac{w(\theta^j_{n-1})}{\sum_{j=1}^m w(\theta^j_{n-1})}$$

where $w(\theta^j_{n-1}) = [d_v \mid v^j_{n-1}, \theta^j_{n-1}]$. The weights $w$ correspond to the inverse gamma distribution in Equation (5) for the observed volume of the $n$-th event conditional on the sampled volumes of the previous event and the remaining parameter, as simulated by the MCMC. The output from the SIR algorithm can be used within Equation (9) to obtain the desired samples of the $n$ interevent time. An brief description of the SIR algorithm is given in Appendix A.
Finally we use the notion of probability gain or information content, as proposed by Kagan and Knopoff, 1987, to make explicit comparisons of different forecasting methods. We calculate the information gain for the present model with respect to other statistical models in the literature. Let A and B be two statistical models, the probability gain is defined as the difference between their log-likelihoods, i.e.:

\[ PG = \sum_{i=m}^{n} \left( l_A(\delta d_{r_i}) - l_B(\delta d_{r_i}) \right). \] (11)

Here \( l_A \) and \( l_B \) are the natural logarithm of the likelihoods for Model A and B respectively and \( m, \ldots, n \) denote the voting phase. These are calculated in a temporal window \( \delta d_{r_i} \) of one month around the observed interevent time in the voting phase. If \( PG \) is greater than zero, Model A has better forecasting performance than Model B, if \( PG \) is zero the two models are equivalent. Together with the total probability gain given by equation (11), we can calculate the “punctual” probability gain, i.e. the probability for each event \( l_A(\delta d_{r_i}) - l_B(\delta d_{r_i}) \) with \( i = m, \ldots, n \) (Passarelli et al, 2010).

3. Application to Kilauea volcano and Mount Etna

We apply the BH_TPM II to Kilauea volcano and Mt Etna eruption data. Marzocchi and Zaccarelli, 2006 have found that Kilauea volcano and Mt. Etna follow a time predictable eruptive behavior. They also stated that these volcanoes are in open conduit regime because of their high eruptive frequency and, consequently, short duration of interevent times. Bebbington, 2007 have showed evidences of the time-predictable character of Mt.
Etna flank eruptions using a catalog starting in 1610 AD. The same results on time-predictability are attained by Sandri et al., 2005 only focusing on the Mt Etna flank eruptions in the period 1971-2002. Passarelli et al., 2010 have found time-predictability of Kilauea volcano for eruptive catalog starting in 1923 AD.

These findings led us to use Kilauea and Mt Etna as test cases for our proposed model. Our goals in this paper is to test: 1) whether or not they follow a time predictable behavior; 2) the reliability of the assumptions used in the model; 3) improvements in using the information given by the volume measurement errors; 4) the ability to fit the observed data, and 5) the forecast capability of the model compared with models previously published in literature for Kilauea and Mt Etna.

3.1. Kilauea volcano

Kilauea volcano is the youngest volcano on the Big Island of Hawaii. The subaerial part of Kilauea is a dome-like ridge rising to a summit elevation of about 1200 m, is about 80 km long, 20 km wide and covers an area of about 1500 km$^2$. Kilauea had a nearly continuous summit eruptive activity during the 19th century and the early part of the 20th century. During the following years, Kilauea’s eruptive activity had shown little change. After 1924, summit activity had become episodic and after a major quiescence period during 1934-1952, the rift activity raised increasing the volcanic hazard (Holcomb, 1987). It is widely accepted that Kilauea has its own magma plumbing system extending from the surface to about 60 km deep in the Earth, with a summit shallow magma reservoir at about 3 km depth. The shallow magma reservoir is an aseismic zone beneath the South zone of the
Kilauea caldera and it is surrounded on two sides by active rift conduits (Klein et al, 1987).

The eruption history of Kilauea volcano directly documented dates back to 18th century, however before the 1923 the eruption record is spotty and in most of the events the erupted volume is unknown. Therefore, we limit our analysis to the 42 events after 1923 AD (please refer to Passarelli et al., 2010 for more details on the Kilauea catalog completeness). The data are listed in Table (1) where we report the onset date of each eruption together with the volume erupted (lava + tephra) and the relative interevent time. The volume of the 1924/05/10 event is taken from http://www.volcano.si.edu/ and is only the tephra volume. Since the eruption that began in 1983 is still ongoing with a volume erupted greater than 3 km$^3$, we have 41 pairs of data of interevent time (i.e. the time between the onset of i-th and the onset of (i+1)-th eruptions) and volume erupted (in the i-th eruption).

In the next two subsections we will present the results of the model for the Kilauea dataset.

3.1.1. Results for variables and parameters

We begin with a discussion of the choice of hyperparameter values. For interevent times we choose an error ($\Delta d_r$) of 1 day for all data in the catalog. For the volumes we assume relative errors ($\Delta d_v/d_v$) of 0.25 for data before the 1960 AD (i.e. $i = 1, \ldots, 13$) and of 0.15 for data after the 1960 AD (i.e. $i = 14, \ldots, 41$) (see discussion in Passarelli et al., 2010). Other hyperparameters for the distributions of $b$ and $c$, are chosen by matching the first two moments of the output of the BH_TPM, i.e. $\mu_b = 0.2$, $\sigma_b = 0.1$, $\mu_c = 200$ days/$10^6 m^3$ and $\sigma_c = 50$ days/$10^6 m^3$ (see Passarelli et al, 2010).
Figure 4).

We run an MCMC simulation for 201,000 iterations with a burn-in of 1,000 iterations and a thinning of one every 20 iterations. We checked the output for convergence and approximate independence of the final sample. In Figure 1 we show the MCMC realizations of $r_i$ and $v_i$ (blue stars), obtained using the whole catalog, and compare with the observed data (red pluses). The plots indicate that the model is able to accurately reproduce the data and that measurement errors have a realistic impact in the estimation uncertainty of the true interevent times and volumes.

Figure 2 shows the posterior distributions of $b$, $c$ and $\lambda$ using all data. As the distribution of $b$ (top left panel) is concentrated within the [0,1] interval, with mean 0.45 and standard deviation 0.05, we infer that the Kilauea volcano has a time predictable behavior. This is compatible with the findings in Passarelli et al. (2010). For the distribution of $c$ (top right panel), which is function of the average magma recharge process, we find that the distribution is mostly contained within the interval $[100, 240]$ days/$10^6$ m$^3$, with mean $164$ days/$10^6$ m$^3$ and a standard deviation $24$ days/$10^6$ m$^3$. In the bottom left panel we have the posterior distribution for $\lambda$, the time of occurrence of the number of events over the length of the catalog. Most of this distribution is contained in the interval $[1.5, 3] \times 10^{-3}$ days$^{-1}$ and has mean $2.0 \times 10^{-3} \times 10^{-3}$ days$^{-1}$ and standard deviation $0.3 \times 10^{-3} \times 10^{-3}$ days$^{-1}$. This results are compatible with the time of occurrence calculated directly from the data with Maximum Likelihood Estimation (MLE) techniques, which yields $\lambda_{MLE} = 1.9 \times 10^{-3} \times 10^{-3}$ days$^{-1}$ with 95 % confidence interval $[1.4, 2.5] \times 10^{-3} \times 10^{-3}$ days$^{-1}$. Figure 3 corresponds to the sequential version of Figure 2. The plots are obtained using the approach discussed in
Figure 1: Blue stars show the posterior distributions of pairs of simulated variables (interevent times $r_i$ and volumes $v_i$). The top panel corresponds to $i = 1, \ldots, 20$ and the bottom panel to $i = 21, \ldots, 41$. 
Figure 2: Posterior distributions for BH TPMII parameters obtained using all data in the catalog: top left panel refers to \( b \), top right to \( c \) and bottom left to \( \lambda \).

The results obtained imply a power law relationship between interevent times and volumes. As discussed in Passarelli et al, 2010, this non linear association underlines the role played by the magma discharging process in the eruption frequency. Such relationship implies the possibility of having a non constant input rate in the magma storage system. Therefore, a large erupted volume may trigger the increasing of the magma upwelling process inside a shallow reservoir. We expect a shorter quiescence period after an eruption characterized by a large volume compared with a process where the magma recharging rate is constant (i.e. classical time predictable model). A simple explanation is the existence of an additional gradient.
of pressure due to the drainage process of the shallow magma system by a large erupted volume. This pressure gradient may increase the magma upwelling process from the deep crust into the shallow storage system. Non constant magma input rate for the shallow magma reservoir for Kilauea volcano has been found by Aki and Ferrazzini (2001) and Takada, (1999). This non-stationarity should be take into account in modeling the magma chamber dynamics at Kilauea volcano.

Figure 3: Posterior distributions of: b parameter in top left panel, c parameter in top right panel and λ in the bottom left panel, all calculated using the sequential procedure discussed in the text. Black dashed line represents the learning phase. Red triangles are the mean of each distribution.
3.1.2. Model checking and Forecasts

We use the ability of our approach to quantify uncertainties in future predictions given the observed data to check the validity of our model. We simulate 10,000 synthetic catalogs using the procedure described in Section 2.5. We then calculate for both, synthetic catalogs and observed data, the rate of occurrence, the maximum, the minimum, the median and the standard deviation. Figure 4 shows the comparisons between the histograms of the synthetic data and the corresponding observed values. Predictions are in good agreement with observed values for the rate of occurrence, the minimum and the median. The are some discrepancies for the maximum and, consequently, for the standard deviation. In these cases the observed value falls in the tails of the predictive distributions. This is due to the fact that the maximum corresponds to the 18 years of quiescence of the Kilauea volcano (i.e. 1934-1952 AD). This is a extraordinary long period of rest for the Kilauea and it could be considered as an extreme value. The second longest interevent time is about 5 years of quiescence (i.e. 1955-1959 AD). Such value falls right at the center of the distribution with $p$-value=0.7. In summary, the model is capable of reproducing the data, with the exception of future extreme events that correspond to the tails of the predictive distribution.

We use the sequential approach of Section 2.5 to evaluate the model’s forecast performance and compare it with published results for the Kilauea volcano’s interevent times. Here we compare our results with those from the homogeneous Poisson process (Klein et al., 1982), the Log-Normal model (Bebbington and Lai, 1996b), the Generalized Time Predictable Model (GTPM) (Sandri et al., 2005) and the BH_TPM (Passarelli et al., 2010).
The homogeneous Poisson implies a totally random and memoryless eruptive behavior. In the Log-Normal model interevent times are described using a log-normal distribution. The mode of a log-normal distribution could reveal a certain degree of ciclicity in the eruptive behavior for Kilauea volcano. The GTPM consists of a linear regression among pairs of interevent times and volumes. The BH_TPM is a hierarchical model where the interevent times and volumes are described via log-normal distributions and uses the logarithm of the generalized time predictable model equation as eruptive behavior.

To gauge the role of the information provided by the volumes in the se-
quential estimation of the interevent times we compare the MCMC samples
of \( \lambda \) with those obtained after the SIR procedure. The results are shown
in Figure 5. From the figure it is clear that the information provided by
the volumes shrinks and shifts the distribution of \( \lambda \). We use the resampled
\( \lambda \) values to calculate the probability gains with respect to the other four
models considered. The results are plotted in Figure 6 where we show the
"punctual" probability gain and we report the total probability gain as cal-
culated using equation (11). As indicated by positive total probability gains
in all cases, our model shows an improvement in forecasting capability when
compared to any of the other four models. The largest gain is observed for
the Poisson model (panel a) where the model provides better forecasts for
20 out of 27 eruptions. The largest global gain is obtained testing against
the GTPM (panel d). This latter results is likely due to the inclusion of
information on measurement error. The smallest overall gain is achieved
with respect to BH_TPM (panel b). This is not surprising as BH_TPM is
the closes model to BH_TPMII among the ones considered.

Overall we observe that BH_TPMII has better forecasting performance
than any of the four competing models in more than 50% of the events. Thus
BH_TPMII seems to be more reliable for probabilistic hazard assessments
that the other models considered.

Finally we investigate possible linear associations between the pointwise
probability gains and the interevent times or volumes in each of the four
considered cases. We only find a significant correlation (\( p \)-value \( \leq 0.01 \)) for
the case of the homogeneous Poisson process. In this case there is a clear in-
verse relationship. This implies that the longer the interevent time the worse
our forecast is. This is justified by the fact that for long quiescence periods
the Kilauea volcano could become memoryless with transition from open to closed conduit regime (see Marzocchi and Zaccarelli, 2006). In addition, considering the events as a point in time (see Bebbington, 2008) together with the fact that we do not consider intrusions not followed by eruptions (Takada, 1999, Dvorak and Dzurisin, 1993) could be distorting. Finally another possible explanation could be related to possible modification of the shallow magma reservoir geometry after an eruption (Gudmundsson, 1986).
3.2. Mount Etna volcano

Mount Etna volcano is a basaltic stratovolcano located in the North-Eastern part of the Sicily Island. It is one of the best known and monitored volcano in the world and records of its activity date back to several centuries B.C. The sub-aerial part of Mount Etna is 3,300 m high covering an area of approximately 1,200 km². Two styles of activity occur at Mt Etna: a quasi-continuous paroxysmal summit activity, often accompanied with explosions,
lava fountains and minor lava emission; a less frequent flank eruptive ac-
tivity, typically with higher effusion rate originate from fissures that open
downward from the summit craters. The flank activity is sometimes ac-
companied by explosions and lava spattering; recently, two flank eruptions
have been highly explosive and destructive, the 2001 and 2002-2003 events

At present there are petrological, geochemical and geophysical evidences
for a 20-30 km deep reservoir controlling the volcanic activity (Tanguy et
al, 1997), but it is still debated whether or not Mt Etna has one or more
shallower plumbing systems. Results from seismic tomography do not reveal
any low velocity zone in the uppermost part of the volcanic edifice, while a
high-velocity body at depth of < 10 km b.s.l. is interpreted as a main solid-
ified intrusive body (Chiarabba et al, 2000, Patanè et al, 2003). However, a
near-vertical shallower plumbing system has been recently inferred at about
4.5 km b.s.l. using deformation data (Bonforte et al, 2008 for a review). It
is widely accepted that a central magma conduit feeds the near-continuous
summit activity, while lateral eruption are triggered by lateral draining of
magma from its central conduit. Only few events appear to be independent
from the central conduit being fed by peripheral dikes (see Acocella and
Neri, 2003 among others).

The recorded eruptive activity for Mt Etna dates back to 1500 B.C.
(Tanguy et al, 2007). Unfortunately, the eruptive catalog can be considered
complete only since 1600 AD for flank eruptions (Mulargia et al, 1985).
Instead summit activity, was recorded carefully only after the World War II
(Andronico and Lodato, 2005) and only after 1970 all summit eruptions were
systematically registered (Wadge, 1975, Mulargia et al, 1987). Thus the Mt
Etna catalog is considered complete since 1970 AD for summit eruptions. There are several catalogs for Mt Etna eruptions available in the literature, the most recent ones being those compiled by Behncke et al (2005), Branca and Del Carlo (2005) and Tanguy et al (2007); the Andronico and Lodato (2005) catalog is detailed only for events in the 20th century. In this study we use only the flank eruptions since 1600 AD using the Behncke et al (2005) catalog as it appears the most complete, at least for volume data. We also integrate and double-check the volume data for the 20th century events with the Andronico and Lodato (2005) catalog. The Behncke et al (2005) catalog lists events up to 2004/09/07 eruption, so we update it for 2006 AD and 2008 AD eruptions using information available in Burton et al (2005) and Behncke et al (2008). A raw estimation for the volume of the 2008/05/13 eruption was kindly provided by Marco Neri (Marco Neri personal communication, 2010).

The choice of using only lateral eruptions needs qualification. Although it could be arguable and could explain only one aspect of the eruption activity at Mt Etna volcano, we are pushed in this direction by the quality of data available. Besides, from a statistical point of view, it is better not to use an incomplete dataset with the awareness of the risk of losing one piece of information, than using incomplete data and find false correlations (Bebbington, 2007). Flank eruptions, however, constitute one of the most important threat for a volcanic hazard assessment at Mt Etna (see Behncke et al, 2005 and Salvi et al, 2006 among others). Thus, in our opinion, the choice of using only flank eruptions seems the best available in a volcanic hazard assessment perspective. In Table 2 the data of flank eruptions at Mt Etna are reported; we indicate the onset date, interevent times ($d_{ri}$) and
volumes \((d_{vi})\). There are 63 eruptive events and consequently 62 pairs of interevent time and volume data.

The next two subsections are organized as follows: we will show first the results obtained for the model parameters both using all data and the sequential procedure discussed in Section 2.5, the ability of the model to fit the data (model checking) and the forecasts obtained. We will compare them with previously published models, when the comparison is possible.

### 3.2.1. Results for variables and parameters

In order to apply the model to the Mt Etna flank eruptions, first we need to specify the measurements errors \((\Delta d_{ri}, \Delta d_{vi})\) and the hyperparameters \((\mu_b, \sigma_b^2, \mu_c\) and \(\sigma_c^2\)) for the priors distribution for \(b\) and \(c\). In the Behncke et al. (2005) catalog there is no mention about the interevent time errors whereas relative errors are given for volume data. Therefore, we assign an error of 1 day for \(\Delta d_{ri}\) for interevent times. According to Behncke et al. (2005) we assign relative errors as follows: \(\Delta d_{vi}/d_{vi} = 0.25\) for \(i = 1, \ldots, 43\), \(\Delta d_{vi}/d_{vi} = 0.05\) for \(i = 44, \ldots, 60\) and \(\Delta d_{vi}/d_{vi} = 0.25\) for \(i = 61, 62\). The latter errors are relative to the 2006 and 2008 AD events not in Behncke et al (2005) catalog; where volumes are the first raw estimate not reparametrized yet (Marco Neri personal communication, 2010). For the hyperparameters we choose the same parameters as the Kilauea case.

The obtained simulations are presented in Figures 7 and 8. As in the Kilauea case, the model reliably reproduces the assumed measurement errors. In Figure 8 we present the results for the model parameters \(b\), \(c\) and \(\lambda\) using all data. As the distribution of \(b\) (top left panel in Figure 8) is within \([0,1]\) with mean and standard deviation \(\bar{b} = 0.30\) and \(\sigma_b = 0.04\) respectively, we
conclude that Mt Etna flank eruptions follow a generalized time predictable eruptive behavior. For the distribution of $c$ (top right panel) we find a value within $[200,460]$ days$/10^6$ m$^3$ with mean $\bar{c} = 330$ days$/10^6$ m$^3$ and error (1 standard deviation) $\sigma_{\bar{c}} = 40$ days$/10^6$ m$^3$. In the bottom left panel we have the posterior distribution for the time of occurrence $\lambda$. This is concentrated in the interval $[3.5,8] \times 10^{-4}$ days$^{-1}$. The mean value and standard deviation are $\bar{\lambda} = 5.4 \times 10^{-4}$ days$^{-1}$ and $\sigma_{\bar{\lambda}} = 0.6 \times 10^{-4}$ days$^{-1}$ respectively. This result is totally compatible with the occurrence time calculated directly by the data with MLE technique, i.e. $\lambda_{MLE} = 4.2 \times 10^{-4}$ days$^{-1}$ with 95% confidence interval $[3.2, 5.4] \times 10^{-4}$ days$^{-1}$. Figure 9 presents the sequential estimation of parameters $b, c$ and $\lambda$.

From the values corresponding to the posterior distributions of $b$ and $c$ we are lead to speculate about the role played by the magma chamber feeding system in the eruption frequency as we have speculated in Section 3.1.1. Mt Etna volcano seems to act as a non-stationary volcano (Mulargia et al, 1987), and the non-stationarity could also imply some sort of cyclicity in the eruption frequency (Behncke and Neri, 2003, Allard et al, 2006). This possible non-stationarity should be taken into account in modeling the magma chamber dynamics at Mt Etna volcano.

3.2.2. Model checking and Forecasts

The results of the model check are presented in Figure 10. It is immediate to realize the agreement of the synthetic simulations (blue bars) with values calculated from the data (red bar) for the rate of occurrence, minimum and median. For the rate of occurrence where the $p$-value=0.94, we can speculate that the model predicts interevent times slightly longer that
the observed one. Although the model works well for minimum, median and rate, it is less satisfactorily for the maximum and, as a consequence, for the standard deviation. For these cases the observed value falls in the tails of the predictive distribution. This can be imputed by the fact that the maximum observed interevent time is relative to a long quiescence period from 1702 to 1755 AD and can be considered an extreme value. By considering the second longest interevent time in catalog, i.e. quiescence period from 1614 AD to 1634 AD, it is compatible with the synthetic maximum distribution with $p$-value=0.7.

Summing up, as for Kilanea data, BH\textsubscript{TPM II} model is able to capture the main data features except for the extreme value that fall within the tail of the predictive distribution.

Using the sequential approach discussed in Section 2.5 now we test the forecast ability of the present model. But, before we embark in this comparison, we present the results of the SIR procedure used to resample the $\lambda^i$’s with the information provided by the erupted volumes. Figure 11 shows the comparison of the MCMC output with the resampled draws. It is clear that the information provided by the volume data in the SIR procedure shrinks and shifts the $\lambda^i$ distributions.

There are several statistical model in literature for the eruptive data series of Mt Etna. The models are: BH\textsubscript{TPM} proposed by Passarelli \textit{et al} (2010); A Non Homogeneous Poisson process with a power law intensity proposed by Salvi \textit{et al} (2006); A Non homogeneous Poisson process with piecewise linear intensity by Smethurst \textit{et al} (2009); the GTPM proposed by Sandri \textit{et al} (2005), and the Hidden Markov Models of Bebbington (2007). The latter model allows the detection of change in volcanic activity using.
Hidden Markov Models. The activity level of Mt Etna volcano is tested through the onset count data, the interevent time data and the quiescence time data (interonset in the Bebbington 2007 terminology) together with time and size-predictable model. Unfortunately, we were not able to apply the sequential procedure to the Bebbington (2007) model due to its intrinsic complexity, so we do not perform the probability gain test against it.

We have already discussed the BH_TPM and GTPM in the previous sections. Salvi et al (2006) proposed a model based on a non homogeneous Poisson process (NHPP). The intensity of the process has a power law time dependence, whose parameters are estimated using MLE. The intensity can increase or decrease with time, depending on the value of the exponent. This provides the ability to fit any trend in eruptive activity. In Smethurst et al (2009), a different (NHPP) was proposed, using a piecewise linear intensity, fitted with numerical MLE. The intensity of the process is constant for eruption before 1970 AD, and then increases linearly with time. The model has a change point that is not easy to handle under our sequential procedure, as the proposed method to estimate it requires the use of all the data. Adding one data point at a time may produce a different estimated change point (see Gasperini et al, 1990). In addition, the estimation of the parameters of the process in the Smethurst et al (2009) model is subject to numerical stability issues that may complicate a sequential approach.

To tackle the change point problem and compute “forward” probabilities of eruptions, we use two different approaches. The first one is to fix the change point (i.e. 1964 AD) at the values estimated in Smethurst et al (2009) and simulate sequentially the other two model parameters.

The second approach consists of empirically estimating the trend for the
process intensity, calculated under the sequential procedure. As we show in Figure 12, after the learning phase, we examine and evaluate the trend for the intensity $\lambda_{MLE}$ (blue stars in the graph), calculated by adding one data at a time, assuming a homogeneous Poisson process. We find that the intensity shows a slow increase with important fluctuations up to the change point found by Smethurst et al. (2009) (black dashed line). Then, after the change point, the intensity rises more markedly. We estimate its trend using linear regressions. In Figure 12, we denote positive significant slopes using green lines. Other cases correspond to red lines. It is clear from the graph that there are no significant trends up to four events after the change point found by Smethurst et al. (2009). This delay in the detection of the change point is due to the sequential nature of the forward procedure. Hence to evaluate probabilities sequentially, we consider an Homogeneous Poisson process up to four events after the change point of Smethurst et al (2009) and then a linearly increasing intensity.
Figure 7: Same as Figure 1. From top to bottom the first panel corresponds to $r_i$ and $v_i$, $i = 1, \ldots, 20$, the second panel corresponds to $i = 21 \ldots 40$ and the third panel corresponds to $i = 40, \ldots, 62$. 
Figure 8: Posterior distributions for BH_TPMII parameters obtained using all data in the catalog: top left panel refers to $b$, top right to $c$ and bottom left to $\lambda$. 
Figure 9: Posterior distributions of: $b$, top panel; $c$, middle panel, and $\lambda$, bottom panel. All distributions are calculated using the sequential procedure discussed in the text. Black dashed lines represent the learning phase. Red triangles correspond to the means.
Figure 10: As Figure 4, histograms of samples from the posterior predictive distributions of several summaries of the interevent times for the Mt Etna (Blue bars). Red dashed lines denote the corresponding observed values. p-values correspond to the proportion of samples above the observed values.
Figure 11: As Figure 5, the SIR procedure is applied to samples of $\lambda$ obtained after the learning phase as required for the sequential approach used (i.e. events from 20 to 62). Blue stars correspond to the MCMC output and red ones to resampled draws.
Figure 12: Trend detection for the intensity of a homogeneous Poisson process using the sequential procedure. Blue stars correspond the intensity calculated sequentially via MLE by adding one data point at a time. Red lines represent non significant regressions (at 1% level), green lines represents significant regressions. The black dashed line is the change point estimated by Smethurst et al 2009. Sequential estimation allows the detection of the change point only four events after the change point found by Smethurst et al., 2009.
Finally we present the results for the probability gain in Figure 13. As it is shown in the inset of each panel, $PG'$s are always greater than zero, showing the present model performs better than the other ones. In particular, the forecasting test against the homogeneous Poisson process (Panel a) shows only 14 eruptions out of 42 with a negative “punctual” probability gain, corroborating the fact that Mt Etna flank eruptions are non-stationary in time (Mulargia et al 1987, Bebbington, 2007, Salvi et al 2006 and Smethurst et al, 2009). In testing against BH_TPM (Panel b), only 17 eruptions have a negative probability gain indicating that modeling Mt Etna interevent times with log-normal distributions does not seem to be the best choice. The result in Panel c against the GTPM is the best one and remarks the limitation of a regression technique in modeling linear relationship between the logarithm of interevent times and of volumes, without using measurement errors. Salvi et al (2006) model, in Panel d, performs worse forecasts compared with BH_TPMII, confirming that a power law intensity is not appropriate for Mt Etna eruption occurrences (Smethurst et al 2009). In Panel e, the test against the Smethurst et al (2009) model, with fixed change point as they found, is the worse one, although the $PG$ is still slightly positive. On one hand, this test shows that modeling the intensity with a linear increasing function for events in the last 40 years seems more appropriate. At the same time, it shows some limitations: a close look at Subplot e shows that event 38 has a very high gain in favor of the BH_TPMII. This event is the 2001 AD eruption, started after 10 years of quiescence. Therefore, the Smethurst et al (2009) model, with the ad hoc fitted piecewise linear intensity, could be misleading for real forecasting purposes as the observed eruption frequency decreases in the
future. Finally we present, in Panel f, the probability gain against the modified Smethurst et al. (2009) model following the specification discussed in the previous paragraph for the “forward” application. Here the probability gain is considerably higher than that in Panel e, although the linear intensity fits better the last part of the catalog.

As a summary, it seems that, the BH_TPMII shows better results in forecasting for more than 50% of the eruptive events manifesting a higher reliability. However, we have to remark that the Smethurst et al. (2009) model is preferable if the Mt Etna flank eruptive frequency keeps increasing in the next years.

We investigate some possible linear relationship between the ”punctual” probability gains and the interevent times or volumes using linear regression analysis. We do not find any correlation between volumes and probability gain. The only significant relationship ($p$-value $\leq 0.01$) is an inverse linear relationship between “punctual” probability gain calculated against the homogeneous Poisson process and interevent times. The inverse relationship implies that we systematically perform worse forecast for long interevent times. We can justify this results stating that for long quiescence periods the volcano becomes memoryless with transition from open and closed conduit regime (see Marzocchi and Zaccarelli, 2006 and Bebbington, 2007).

Another explanation could be related to the complexity of the volcano eruption system not considered in this model. The time predictable model seems more appropriate when the eruptions are close in time. Conversely, when the quiescence period are extremely long, other compelling physical processes may control the volcanic activity. Finally, by neglecting the summit activity we lose one piece of information related to the amount of erupted
Figure 13: “Punctual probability gain” of the BH_TPMII for each event after the learning phase with respect to: Poisson Model (Klein, 1982) (Panel a); BH_TPM (Passarelli et al, 2010) (Panel b); GTPM (Sandri et al, 2005) (Panel c); Salvi et al. 2006 (Panel d); Smethurst et al. 2009 (Panel e); Modified piecewise linear model of Smethurst et al, 2009 (Panel f). Values greater than zero indicate that BH_TPM model performs better than the reference models. The inset in each panel is the total Probability gain.

volume from summit crater during the quiescence period. This may introduce a bias that could explain the inverse relationship.

4. Conclusion

In this work we propose a Bayesian Hierarchical model to fit a time predictable model for open conduit volcanoes (BH_TPMII). The use of Bayesian Hierarchical model provides a suitable tool to take into account
the uncertainties related to the eruption process as well as those relative
to the data, parameters, and variables. We have applied the model to the
Kilauea eruptive catalog from 1923 to 1983 AD and to Mount Etna flank
eruptions from 1607 to 2008 AD. The results show that both volcanoes have
a time predictable eruptive behavior where interevent times depend on the
previous volume erupted. The numerical values of the time predictable
model parameters inferred, suggest that the amount of the erupted volume
could change the dynamics of the magma chamber refilling process during
the repose period.

The model shows a good fit with the observed data for both volcanoes
and is also able to capture extreme values as a tail behavior of the dis-
tributions. The forecasts obtained by BH_TPM II are superior to those
provided by other statistical models for both volcanoes. In particular we
have improved the forecast performance compared to that of BH_TPM. It
is important to notice that a model based on a NHPP, as the one developed
in Smethurst et al (2009), could provide better forecast if the flank eruptive
activity on Mt Etna keeps increasing in time in the same fashion as it did
in the last 40 years; any change from this trend may cause wrong forecasts
of the Smethurst et al’s (2009) model. Finally, we remark again that the
model proposed here may be used for real prospective long-term forecasts
to Kilauea and Mount Etna volcano.
A. Sampling Importance Resampling algorithm

The Sampling Importance Resampling (SIR) is a non iterative procedure proposed by Rubin (1988). The SIR algorithm generates an approximately independent and identically distributed (i.i.d.) sample of size $m$ from the target probability density function $f(x)$. It starts by generating $M$ ($m \leq M$) random numbers from a probability density function $h(x)$ as inputs to the algorithm. The output is a weighted sample of size $m$ drawn from the $M$ inputs, with weights being the importance weights $w(x)$. As expected, the output of the SIR algorithm is good if the inputs are good ($h(x)$ is close to $f(x)$) or $M$ is large compared to $m$.

The SIR consists of two steps: a sampling step and an importance resampling step as given below:

1. (Sampling step) generate $X_1, \ldots, X_M$ i.i.d. from the density $h(x)$ with support including that of $f(x)$;

2. (Importance Resampling Step) draw $m$ values $Y_1, \ldots, Y_m$ from $X_1, \ldots, X_M$ with probability given by the importance weights:

$$w^*(X_1, \ldots, X_M) = \frac{w(X_i)}{\sum_{j=1}^{M} w(X_j)} \quad \text{for} \quad i = 1, \ldots, M.$$ 

where $w(X_j) = f(X_j)/h(X_j)$ for all $j$.

The resampling procedure can be done with or without replacement.
References


Table 1: Catalog of eruptive events at Kilauea volcano

<table>
<thead>
<tr>
<th>Eruption #</th>
<th>Onset yyyymmdd</th>
<th>Interevent time [days]</th>
<th>Volume lava e tephra $[10^6 m^3]$</th>
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</thead>
<tbody>
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<td>1</td>
<td>1923 08 25</td>
<td>259</td>
<td>0.073</td>
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