CMPS 201
Midterm 2
Review Problems

Study problems 2 and 3 on the midterm 1 review sheet, as well as the posted solutions to homework assignments 4, 5 and 6.

1. Recall the n^{th} harmonic number was defined to be $H_n = \sum_{k=1}^{n} \left(\frac{1}{k}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$. Use induction to prove that

$$\sum_{k=1}^{n} kH_k = \frac{1}{2}n(n+1)H_n - \frac{1}{4}n(n-1)$$

for all $n \geq 1$. (Hint: Use the fact that $H_n = H_{n-1} + \frac{1}{n}$, and note this was a previous hw problem.)

2. Recall that the average case runtime $t(n)$ of the i^{th} selection problem satisfies the recurrence

$$t(n) = (n-1) + \left(\frac{n-1}{n^2}\right)\sum_{q=1}^{n-1} t(q)$$

Use this recurrence to prove that $t(n) = \Theta(n)$. (Hint: See problem 3 on the midterm 1 review sheet.)

3. Suppose we are given 4 gold bars (labeled 1, 2, 3, 4), one of which *may* be counterfeit: gold-plated tin (lighter than gold) or gold-plated lead (heavier than gold). Again the problem is to determine which bar, if any, is counterfeit and what it is made of. The only tool at your disposal is a balance scale, each use of which produces one of three outcomes: tilt left, balance, or tilt right.

   a. Use a decision tree argument to prove that at least 2 weighings must be performed (in worst case) by any algorithm that solves this problem. Carefully enumerate the set of possible verdicts.
   b. Determine an algorithm that solves this problem using 3 weighings (in worst case). Express your algorithm as a decision tree.
   c. Find an adversary argument that proves 3 weighings are necessary (in worst case), and therefore the algorithm you found in (b) is best possible. (Hint: study the adversary argument for the min-max problem discussed in class to gain some insight into this problem. Further hint: put some marks on the 4 bars and design an adversary strategy that, on each weighing, removes the fewest possible marks, then show that if the balance scale is only used 2 times, not enough marks will be removed.)

4. Let $B = b_1 b_2 \ldots b_n$ be a bit string of length n. Consider the following problem: determine whether or not B contains 3 consecutive 1's, i.e. whether B contains the substring "111". Consider algorithms that solve this problem whose only allowable operation is to peek at a bit.

   a. Suppose $n = 4$. Obviously 4 peeks are sufficient. Give an adversary argument showing that in general, 4 peeks are also necessary. (Hint: this problem is similar to problem 4b on hw6, and has a similar solution.)
   b. Suppose $n \geq 5$. Give an adversary argument showing that $4 \cdot \lfloor n/5 \rfloor$ peeks are necessary. (Hint: divide B into $\lfloor n/5 \rfloor$ 4-bit blocks separated by 1-bit gaps between them. Thus bits 1-4 form the first block, and bit 5 is the first gap. Bits 6-9 form the next block and bit 10 is the next gap, etc.. Any leftover bits form a separate block. Now run the adversary from part (a) on each of the 4-bit blocks.)