CMPS 201
Midterm 1
Review Problems

1. Let $T(n)$ satisfy the recurrence $T(n) = aT(n/b) + f(n)$, where $a \geq 1$, $b > 1$ and $f(n)$ is a polynomial satisfying $\deg(f) > \log_b(a)$. Prove that case (3) of the Master Theorem applies, and in particular, prove that the regularity condition necessarily holds.

2. The n^{th} harmonic number is defined to be $H_n = \sum_{k=1}^{n}\left(\frac{1}{k}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$. Use induction to prove that
$$\sum_{k=1}^{n} H_k = (n+1)H_n - n$$
for all $n \geq 1$. (Hint: Use the fact that $H_n = H_{n-1} + \frac{1}{n}$.)

3. Define the sequence S_n by the recurrence $S_n = (n-1) + \frac{n-1}{n^2} \cdot \sum_{k=1}^{n-1} S_k$. Use induction to prove $S_n \leq 2n$ for all $n \geq 1$.

4. The following sorting algorithm, called BadSort() is a modified version of StoogeSort() from the 2nd edition of CLRS, which seems to have been left out of the 3rd edition.

```
BadSort(A, p, r)   pre: p ≤ r
if  A[p] > A[r]
     A[p] ↔ A[r]  (swap)
if  p + 1 ≥ r
     return
else
     q = ⌊(r − p + 1)/3⌋
     BadSort(A, p, r − q)
     BadSort(A, p + q, r)
     BadSort(A, p, r − q)
```

   a. Use induction on the length $m = r - p + 1$ of $A[p \cdots r]$ to prove the correctness of BadSort().
   b. Write a recurrence relation for the number of array comparisons performed by BadSort() on an array of length n.
   c. Use the Master Theorem to find an asymptotic solution to this recurrence, and explain what is bad about BadSort().

5. Simplify the recurrence for MergeSort() by assuming that n is an exact power of 2; $n = 2^k$ for some integer $k \geq 0$.
$$T(n) = \begin{cases} 0 & n = 1 \\ 2T\left(\frac{n}{2}\right) + (n-1) & n \geq 2, n = 2^k \end{cases}$$
Use the iteration method to find an exact solution to this recurrence.

6. Write a recursive algorithm (modeled on MergeSort()) that determines if an array is sorted, i.e. given an array $A = (A_1, A_2, \dots, A_n)$ as input, return TRUE/FALSE iff A is/is-not arranged in increasing order. Prove the correctness of your algorithm. Write a recurrence for the number $T(n)$ of array comparisons performed by your algorithm. Check that $T(n) = n - 1$ is the exact solution to your recurrence.

7. Given $A = (A_1, A_2, \dots, A_n)$, a pair of indices (i, j) is called an *inversion* iff both $i < j$ and $A_i > A_j$. Write a recursive algorithm that determines the number of inversions in its input array A. Do this in such a way that the worst case number of comparisons performed is $T(n) = \Theta(n \log n)$. (Hint: modify MergeSort() so that it counts inversions as it sorts.)