Chapter 7

Directed Graphs

7.1 Definitions (and More Definitions)

Consider the set of boys Alan, Bill, Charlie and Don and the set of girls Ethel, Florence and Gail. Alan cares only for Ethel, Bill and Charlie are both keen on Florence, but Charlie is also interested in Gail. Don is a confirmed bachelor and has nothing to do with any of the girls. Ethel's heart belongs to Alan, Gail fancies Don and Florence doesn't care for any of them. We can represent this by the diagram in Figure 7.1. Here we have drawn an arrowed line **from** a point x **to** a point y, i.e., the arrow points from x to y, if x "is fond of" y.

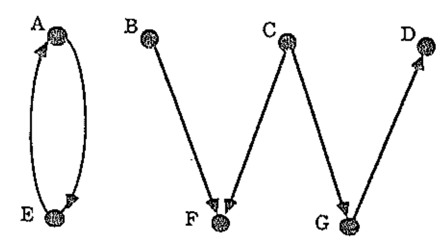

Figure 7.1: A fondness diagram.

Another way of describing this "being fond of" relationship is to list it as a set of **ordered** pairs:

$$(A,E),\ (B,F),\ (C,F),\ (C,G),\ (E,A),\ (G,D)$$

where (x,y) is in the list to signify that x is fond of y. The order in the pair (x,y) is important—X first, Y second, so that (x,y) is different from (y,x) if x is different from y. Hence the term **ordered** pair. In Figure 7.1 this ordering was represented by the direction of the arrow head.

This brings us to the concept of a directed graph.

A **directed graph** $D = (V, A)$ consists of two finite sets

V, the **vertex set**, a nonempty set of elements called the **vertices** of D and A, the **arc set**, a (possibly empty) set of elements called the **arcs** of D,

such that each arc a in A is assigned an **ordered** pair of vertices (u, v).

If a is an arc, in the directed graph D, with associated ordered pair of vertices (u, v), then a is said to **join** u **to** v, u is called the **origin** or the **initial vertex** or the **tail of** a, and v is called the **terminus** or the **terminal vertex** or the **head of** a.

We represent directed graphs diagrammatically just as for graphs except that arcs are represented by *arrowed* lines. Thus, for example, Figure 7.2 represents the directed graph D with vertex set $V = \{u, v, w, x\}$ and arc set $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ where

a_1 joins u to v and so has associated ordered pair (u, v),
a_2 joins v to v and so has associated ordered pair (v, v),
a_3 joins v to w and so has associated ordered pair (v, w),
a_5 joins w to v and so has associated ordered pair (w, v), and so on.

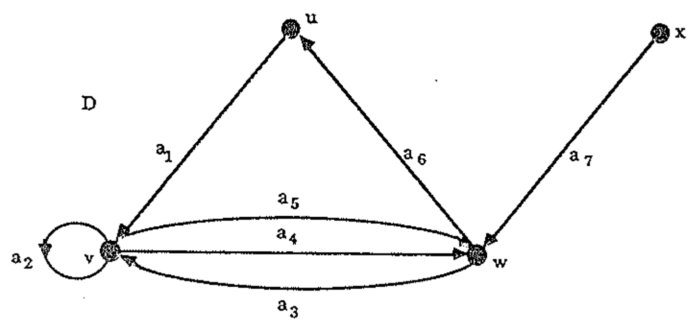

Figure 7.2: A digraph.

We shall abbreviate the term directed graph to **digraph** and also sometimes speak of arcs as **directed edges** (or simply **edges**).

If at some time more than one digraph is being considered we may denote the vertex set of a digraph D by $V(D)$ and similarly the arc set by $A(D)$.

Given a digraph D we can obtain a **graph** G from D by "removing all the arrows" from the arcs. More formally this graph G has the same vertex set as D and corresponding to each arc a in D with associated ordered pair of vertices (u, v) there is an edge e in G with associated *unordered* pair (u, v). G is then called the **underlying graph** of D.

Thus Figure 7.3 shows the underlying graph of the digraph D of Figure 7.2.

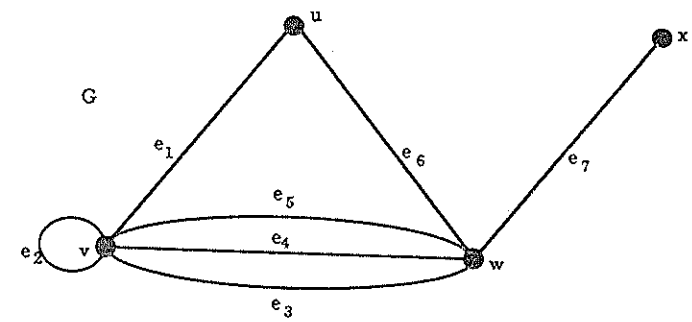

Figure 7.3: The underlying graph of the digraph of Figure 7.2.

Many of the definitions that we gave for graphs have analogues for digraphs. We now give some of these.

Let D be a digraph. Then a **directed walk** in D is a finite sequence

$$W = v_0 a_1 v_1 \ldots a_k v_k,$$

whose terms are alternately vertices and arcs such that for $i = 1, 2, \ldots, k$, the arc a_i has origin v_{i-1} and terminus v_i.

As in graphs, this directed walk W is often written simply as its sequence of vertices

$$W = v_0 v_1 \ldots v_k,$$

the number k of arcs in W is called the **length** of W, and we can have directed walks of length 0 where the sequence consists solely of one vertex (and no arcs), for example $W = v_0$.

There are similar definitions for **directed trails, directed paths, directed cycles** and **directed tours**.

For example, in the digraph D of Figure 7.2,

$$W = u\ a_1\ v\ a_2\ v\ a_3\ w\ a_5\ v\ a_3\ w$$

is a directed walk of length 5. It is not a directed trail since the arc a_3 occurs twice. Also

$$T = x\ a_7\ w\ a_5\ v\ a_2\ v\ a_3\ w$$

is a directed trail (of length 4). It is not a directed path since the vertex v occurs twice (as does the vertex w). Similarly

$$P = x\ a_7\ w\ a_6\ u\ a_1\ v$$

is a directed path, while

$$C = v\ a_4\ w\ a_6\ u\ a_1\ v$$

is a directed cycle.

The walk W given in the definition above is said to be a $v_0 - v_k$ **walk** or a **walk from** v_0 **to** v_k. The vertex v_0 is called the **origin** of the walk W, while v_k is called the **terminus** of W.

A vertex v of the digraph D is said to **reachable** from a vertex u if there is a directed path in D from u to v.

In the digraph D of Figure 7.2, u, v and w are all reachable from x but x is not reachable from any vertex apart from itself.

In digraphs there are two natural notions of connectedness.

A digraph D is said to be **weakly connected** or **connected** if its underlying graph is connected.

Thus our example D of Figure 7.2 is weakly connected.

A digraph D is said to be **strongly connected** (or **diconnected**) if for any pair of vertices u and v in D there is a directed path from u to v, i.e., given any pair of vertices in D, each is reachable from the other.

Since x is not reachable from any other vertex, the digraph of Figure 7.2 is not strongly connected. However the digraph of Figure 7.4 *is* strongly connected.

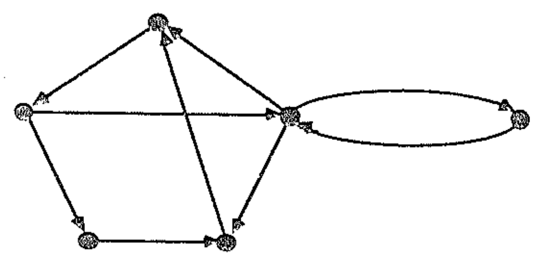

Figure 7.4: A strongly connected digraph.

Given a *graph* G we can obtain a digraph from G by specifying for each edge in G an order to its end vertices. Such a digraph D is called an **orientation** of G.

The graph K_3 has eight different orientations, shown as $D_1, \ldots, D_8$ in Figure 7.5, although, of these, only two are strongly connected, namely D_1 and D_8.

Two digraphs D_1 and D_2 are said to be **isomorphic** if there is a one-to-one and onto correspondence between $V(D_1)$ and $V(D_2)$ and also a one-to-one and onto correspondence between $E(D_1)$ and $E(D_2)$ such that if arc e_1 in D_1 goes from vertex u_1 to v_1 then the corresponding arc e_2 in D_2 goes from vertex u_2 to vertex v_2 where u_2 and v_2 are the vertices in D_2 corresponding to u_1 and v_1 respectively. Of course, such a pair of correspondences is called a (digraph) **isomorphism**.

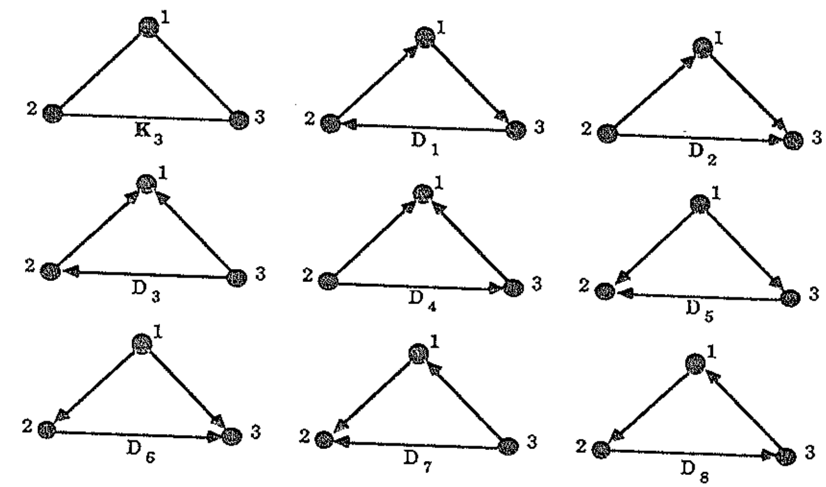

Figure 7.5: The eight orientations of K_3.

It can be easily seen that, among the orientations of K_3 shown in Figure 7.5, D_1 and D_8 are isomorphic with a suitable vertex correspondence given by

$$1 \sim 1,\ 3 \sim 2,\ 2 \sim 3,$$

while $D_2, D_3, \ldots, D_7$ are all isomorphic to each other. For example, for D_2 and D_6 the vertex correspondence is

$$1 \sim 2,\ 2 \sim 1,\ 3 \sim 3.$$

A **strong component** S of a digraph D is a subdigraph of D which is strongly connected and is not a proper subdigraph of any other strongly connected subdigraph of D.

This notion corresponds to that of connected components in graphs. Using arguments similar to that used for graphs we can show that if the vertex u belongs to the strong component S of the digraph D then S consists of all those vertices v of D such that v is reachable from u and u is reachable from v together with all the arcs in D which join such vertices. (We realise that we have not formally defined the term **subdigraph** used in the definition but feel confident in leaving this to the reader.) Figure 7.6 shows a digraph D and its four strong components.

A digraph D is called **simple** if, for any pair of vertices u and v of D, there is at most one arc from u to v and there is no arc from u to itself.

For example, the digraph D of Figure 7.2 is not simple but that of Figure 7.4 is.

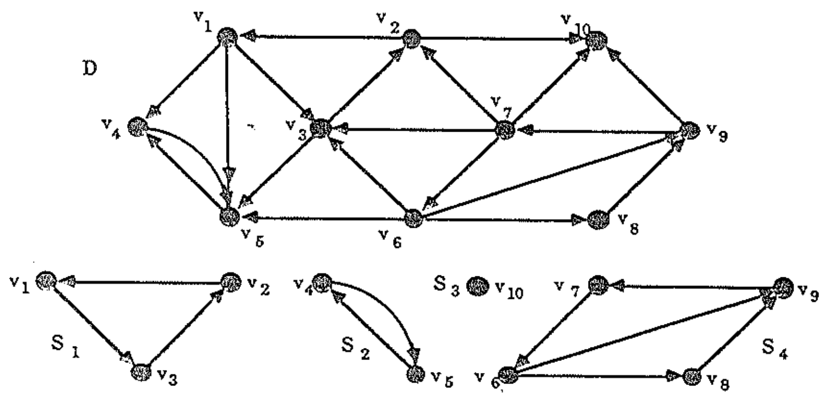

Figure 7.6: A digraph and its strong components.

Exercises for Section 7.1

7.1.1 Let D be the digraph of Figure 7.7.

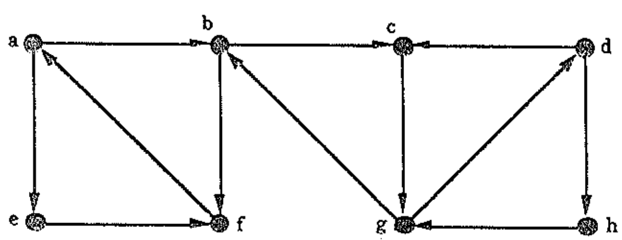

Figure 7.7

(a) Find a directed walk in D of length 8. Is this walk a directed path?

(b) Find a directed trail in D of length 10.

(c) Find a directed path in D of longest possible length.

(d) Find a directed cycle in D of longest possible length.

(e) Is D weakly connected?

(f) Is D strongly connected? If so, give an example of a directed path from u to v for each pair of vertices u and v of D. If D is not strongly connected, find a pair of vertices u and v such that u is not reachable from v.

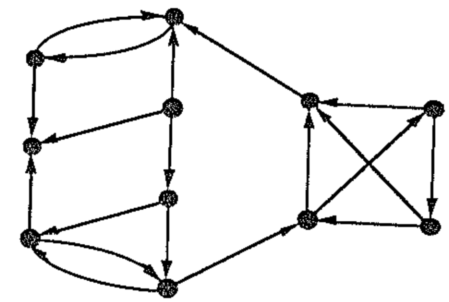

Figure 7.8

7.1.2 Find the strong components of the digraph of Figure 7.8.

7.1.3 A digraph D is said to be **unilaterally connected** if, given any pair of vertices u and v of D, either u is reachable from v or v is reachable from u, but not necessarily both.

(a) Show that the digraph of Figure 7.9 is unilaterally connected but not strongly connected.

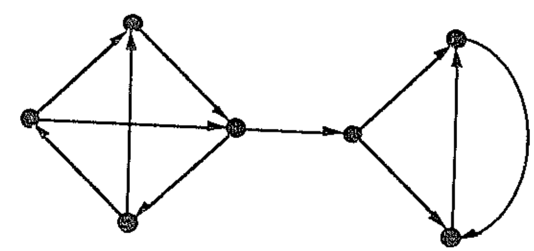

Figure 7.9: A unilaterally connected digraph which is not strongly connected.

(b) Determine whether or not the digraph of Figure 7.8 is unilaterally connected.

(c) Prove that a digraph D is unilaterally connected if and only if there is a directed walk in D involving all the vertices of D, i.e., D has a directed **spanning** walk.

7.1.4 Let u and v be distinct vertices of the digraph D. Prove that every directed $u - v$ walk in D contains a directed $u - v$ path. (This is the digraph analogue of Theorem 1.3.)

7.1.5 Prove that a digraph D is strongly connected if and only if it has a closed directed walk going through all its vertices. (Compare this result with Exercise 7.1.3 (c).)

7.1.6 Which pairs of the digraphs given in Figure 7.10 are isomorphic? For those pairs that are isomorphic, write down an isomorphism. For those pairs that are not isomorphic, explain clearly why they are not.

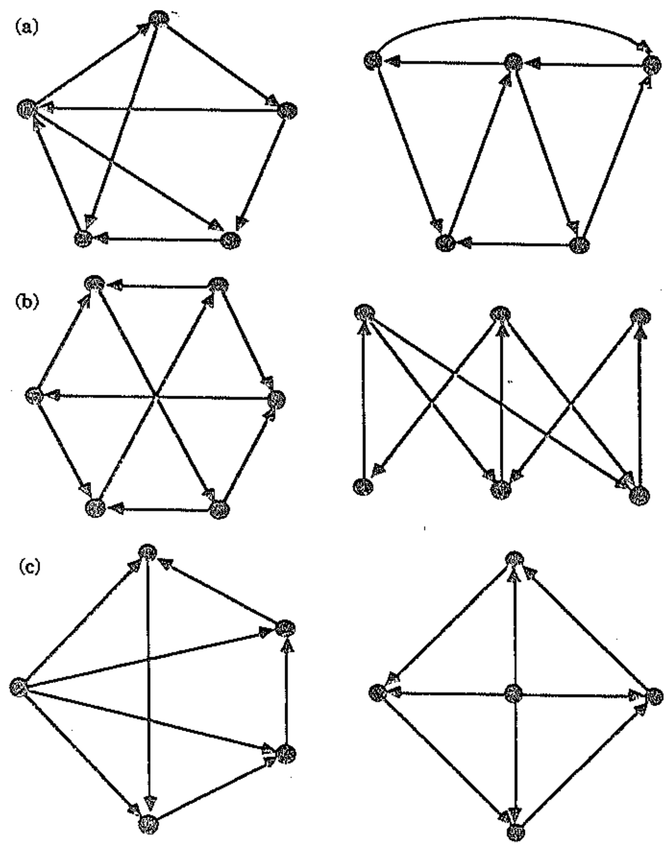

Figure 7.10: Pairs of isomorphic digraphs?

7.1.7 Let D be a simple digraph. We define the **complement** of D to be the simple digraph $\overline{D}$ with the same vertex as D and where there is an arc from a vertex u to a vertex v if and only if there is no such arc in D. Figure 7.11 illustrates a digraph and its complement.

Give an example of a simple weakly connected digraph D which is not unilaterally connected but its complement is

(a) strongly connected,

(b) unilaterally connected but not strongly connected.

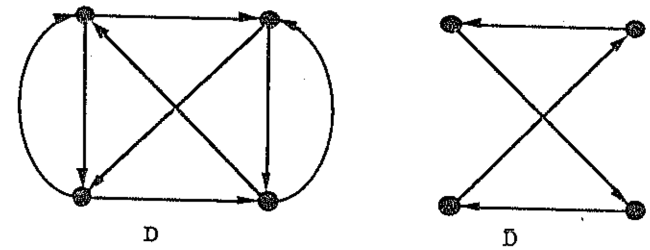

Figure 7.11: A simple digraph and its complement.

7.1.8 Given a digraph D we define its **condensation** to be the simple digraph D^* whose vertices $v_1, \ldots, v_n$ are in one-to-one correspondence with the strong components $S_1, \ldots, S_n$ of D and such that there is an arc a from u_i to u_j (for $i \neq j$) if and only if there is an arc from some vertex of S_i to some vertex of S_j. Figure 7.12 shows a digraph D and its condensation D^*.

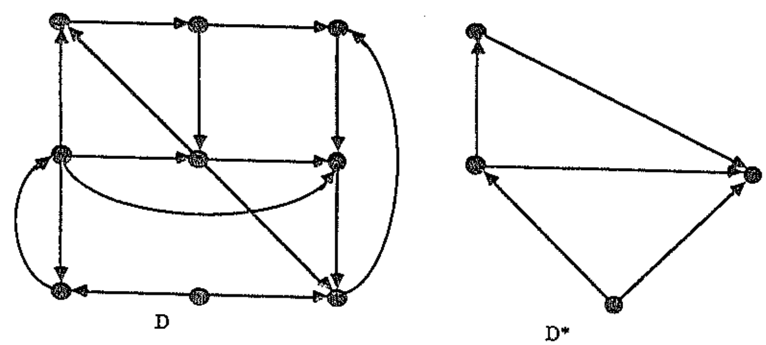

Figure 7.12: A digraph and its condensation.

(a) Find the condensation D^* of the digraph D of Figure 7.8.

(b) Prove that the condensation D^* of any digraph D has no directed cycles.

(c) Prove that the condensation D^* of any digraph D is strongly connected, unilaterally connected or weakly connected if and only if D is strongly connected, unilaterally connected or weakly connected, respectively. (This shows that the condensation construction produces a digraph D^*, usually much simpler than the given digraph D, with the same connectedness attributes as D.)

7.1.9 Construct the digraph D' from a given digraph D by introducing a new vertex v and an arc joining v to each vertex of D and an arc joining each vertex of D to v. Prove that D' is strongly connected.

7.1.10 The **converse** of a digraph D, denoted by $\overleftrightarrow{T}$, is the digraph obtained from D by reversing the direction of each arc of D.

(a) Prove that a vertex v is reachable from u in D if and only if u is reachable from v in $\overleftrightarrow{T}$.

(b) Prove that D is weakly connected, unilaterally connected or strongly connected if and only if $\overleftrightarrow{T}$ is weakly connected, unilaterally connected or strongly connected, respectively.

(c) Show that for every $n \geq 1$ there is a digraph D on n vertices which is isomorphic to its converse.

7.2 Indegree and Outdegree

Let v be a vertex in the digraph D. The **indegree** $id(v)$ of v is the number of arcs of D that have v as its head, i.e., the number of arcs that "go to" v. Similarly, the **outdegree** $od(v)$ of v is the number of arcs of D that have v as its tail, i.e., that "go out" of v.

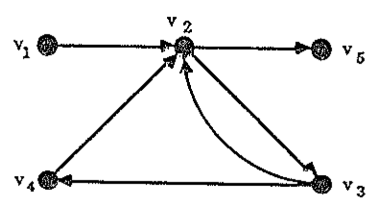

Figure 7.13

Thus in the digraph of Figure 7.13 we have $id(v_1) = 0$, $id(v_2) = 3$, $id(v_3) = 1$, $id(v_4) = 1$, $id(v_5) = 1$, while $od(v_1) = 1$, $od(v_2) = 2$, $od(v_3) = 2$, $od(v_4) = 1$, $od(v_5) = 0$.

Our first theorem on digraphs is the analogue of Theorem 1.1 on graphs.

Theorem 7.1 (The First Theorem of Digraph Theory) *Let D be a digraph with n vertices and q arcs. If $\{v_1, \ldots, v_n\}$ is the set of vertices of D then*

$$\sum_{i=1}^{n} id(v_i) = \sum_{i=1}^{n} od(v_i) = q.$$

Proof When the indegrees of the vertices are summed, each arc is counted exactly once since every arc goes to exactly one vertex. Thus

$$\sum_{i=1}^{n} id(v_i) = q.$$

Similarly, when the outdegrees are summed, each arc is counted exactly once since every arc goes out of exactly one vertex and this gives the other equality. □

We continue with our graph theory analogies.

Let D be a weakly connected digraph. Then a **directed Euler trail** in D is a directed open trail of D containing all the arcs of D (once and only once). A **directed Euler tour** of D is a directed closed trail of D containing all the arcs of D (once and only once). A digraph D containing a directed Euler tour is called an **Euler digraph**.

For example, in Figure 7.14 the digraph D_1 is Euler with directed Euler tour

$$a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9\ a_{10}\ a_{11}$$

while the digraph D_2, although it is not Euler, has a directed Euler trail, e.g.,

$$a_1\ a_2\ a_3\ a_4\ a_5.$$

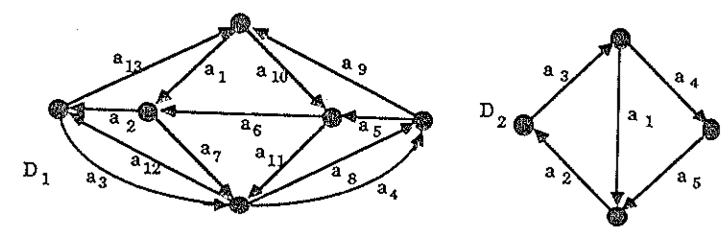

Figure 7.14: D_1 is an Euler digraph and D_2 has a directed Euler trail.

We now give a characterisation of Euler digraphs which is very similar to that of Euler graphs given in Theorem 3.2.

Theorem 7.2 *Let D be a weakly connected digraph with at least one arc. Then D is Euler if and only if $od(v) = id(v)$ for every vertex v of D.*

Proof Suppose first that D is Euler and let T denote a directed Euler tour of D, beginning (and so finishing) at the vertex v. Then, if u is a vertex different from v, each time that u is encountered on T it is entered by an arc and left by an arc and so each occurrence of u in T represents a contribution of 1 to the indegree of u and 1 to the outdegree of u. Since every arc incident with u occurs in T it follows that

$od(u) = id(u)$. It is left to consider the initial vertex v of T. Since T begins and ends at v the first arc of T contributes 1 to the outdegree of v while the last arc contributes 1 to its indegree. Since every other occurrence of v on the tour T contributes 1 each to its outdegree and indegree, it follows once again that $od(v) = id(v)$.

For the converse, suppose that D is a weakly connected digraph with $od(v) = id(v)$ for every vertex v of D. We use induction on q, the number of arcs in D. To begin with, if D has only one arc, a, say, with head u and tail v, then $u = v$ since otherwise either $od(v) = 1$ but $id(v) = 0$ or $od(u) = 1$ but $id(u) = 0$, in contradiction to the hypothesis on D. Thus a is a directed loop and by itself gives a directed Euler tour of D.

Although it is not necessary for us to do this, let us now suppose that $q = 2$. Then, again by the hypothesis, either both of the two arcs are loops incident with the same vertex or they are both non-loops with opposite heads and tails as shown in Figure 7.15. Clearly both the resulting digraphs have a directed Euler tour.

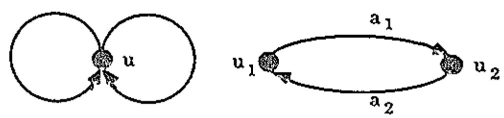

Figure 7.15: The only Euler digraphs with two arcs.

Now assume that q is a fixed number with $q \geq 3$, and that all weakly connected digraphs with less than q arcs in which every vertex has equal outdegree and indegree are Euler. Since our digraph D is weakly connected and $od(v) = id(v)$ for every vertex v, every vertex of D has positive outdegree, i.e., its outdegree is not zero.

Now select any vertex u in D. Since $od(u) > 0$ there exists a trail W' in D starting at u. If W' also finishes at u then we have a closed $u - u$ trail in D. If W' finishes at $v \neq u$ then, using the "contribution argument" of the first part of the proof there must be an arc a in D going out of v which is not part of W' and so we can extend W' to a longer trail. Clearly we can only make these extensions a finite number of times before we are forced to finish the trail at u, our starting place. This argument shows that we can find a closed trail W starting (and so finishing) at the vertex u.

Now if this closed trail W contains every arc of D we are finished since it is a directed Euler tour and so D is Euler. Otherwise there are arcs of D that do not belong to W. Remove from D all those arcs in W together with any resulting isolated vertices to obtain a new digraph F. Since for every vertex v of W we have $od_W(v) = id_W(v)$ (where $od_W(v)$ and $od_W(v)$ denote the outdegree and indegree, respectively, of v *in* W) it follows that *in* F $od(v) = id(v)$ for every v in F. Now the connected components of the underlying graph G of F produce weakly connected subdigraphs of F each having less than q edges and with every vertex having equal indegree and outdegree. Thus, by our induction hypothesis, each such subdigraph is Euler. Moreover, since D is weakly connected, each of these subgraphs has a vertex in common with W. A directed Euler tour can now be constructed by attaching to W at each of these common vertices the Euler tour of the subdigraph. □

The following is the digraph analogue of Theorem 3.3.

Theorem 7.3 *Let D be a weakly connected digraph with at least two vertices. Then D has a directed Euler trail if and only if D has two vertices u and v such that*

$$od(u) = id(u) + 1 \quad and \quad id(v) = od(v) + 1$$

and, for all other vertices w of D, $od(w) = id(w)$. Furthermore, in this case the trail begins at u and ends at v.

Proof Suppose that D contains an Euler trail W that begins at u and ends at v. Then, as in the first part of the proof of Theorem 7.2, for every vertex w different from both u and v we get $od(w) = id(w)$. Moreover the first arc of W contributes 1 to the outdegree of u while every other occurrence of u in W contributes 1 each to the outdegree of u and the indegree of u. Therefore $od(u) = id(u) + 1$. Similarly $id(v) = od(v) + 1$.

Conversely, let D be a weakly connected digraph containing vertices u, v as described in the statement and with $od(w) = id(w)$ for all other vertices w. Add a new arc a to D joining v to u. This produces a new digraph F in which the new outdegree of v is one more than its old outdegree so now $id(v) = od(v)$, similarly $id(u) = od(u)$, and for every other vertex w we still have $id(w) = od(w)$. Moreover F is weakly connected and so since $id(x) = od(x)$ for every vertex x in F it follows from Theorem 7.2 that F is Euler. Let T then be a directed Euler tour. Then T contains all the arcs of D together with the added arc a. Deleting this arc a produces a directed Euler trail back in our digraph D, and this trail must start at u and finish at v, as required. □

We can apply Theorems 7.2 and 7.3 to the digraphs D_1 and D_2 respectively of Figure 7.14 to verify, just by looking at the outdegrees and indegrees of the vertices, that they have a directed Euler tour and trail respectively (and in D_2 the trail must start at the tail of the arc a_1 and finish at the head of the arc a_2).

We now discuss an application of directed Euler tours to a problem in coding theory. Let $\Sigma = \{0, \ldots, n-1\}$ be an alphabet of n letters. We can form precisely n^k different sequences of length k using these letters. Such a sequence is called a **word** of length k from Σ.

> An (n, k) **de Bruijn sequence** is a sequence
>
> $$a_0 a_1 \ldots a_{t-1}$$
>
> of letters from $\Sigma = \{0, \ldots, n-1\}$ such that every word w of length k from Σ can be written in the form
>
> $$w = a_i a_{i+1} \ldots a_{i+k-1}$$
>
> for a unique $i \in \{0, \ldots, t-1\}$. (In the case where $i \geq t - 1 - k$ this form is interpretted to be
>
> $$a_i a_{i+1} \ldots a_{t-1} a_0 \ldots a_{i+k-t-1},$$
>
> so that, in effect, the sequence $a_0 a_1 \ldots a_{t-1}$ is cyclical.)

For example, consider the case alphabet $n = 2$, so that $\Sigma = \{0,1\}$. Then the sequence

$$1\,0\,0\,0\,1\,1\,1\,0$$

is a $(2,3)$ de Bruijn sequence. To see this, take sections of three consecutive letters $a_j a_{j+1} a_{j+2}$ from the sequence, starting at its beginning (and running over to the beginning again for the last two words). This produces all of the $2^3 = 8$ different words of length 3 from Σ namely:

$$100, 000, 001, 011, 111, 110, 101, 010.$$

In general the process of producing all of the n^k words from a given (n,k) de Bruijn sequence is to take the first k letters from the sequence for the first word, then "shift" this along by one place to get the second word and continue this shifting procedure until we come full circle back to the first word. Since there are n^k shifts in all, we see that any (n,k) de Bruijn sequence has precisely n^k terms.

De Bruijn sequences are very important in coding theory and are used by shift registers in the case where $\Sigma = \{0,1\}$. For this reason they are also called **shift register sequences.**

We now associate to any given (n,k) de Bruijn sequence a digraph $D_{n,k}$, called a **de Bruijn diagram** or **Good diagram**. The vertex set V of $D_{n,k}$ is defined to be the set of all words of length $k-1$ from the alphabet $\Sigma = \{0,\ldots,n-1\}$, so that $D_{n,k}$ has n^{k-1} vertices. We now introduce an arc from each such vertex (word) $b_1 b_2 \ldots b_{k-1}$ to each vertex (word) of the form $b_2 b_3 \ldots b_k$, (so that we cancel off the first term of the tail of the arc and add on a new last term to get the head of the arc). We label this arc unambiguously by $b_1 b_2 \ldots b_n$. Note that there are n^k such arcs, and that each corresponds uniquely to an (n,k) de Bruijn sequence. Figures 7.16 and 7.17 illustrate the two digraphs $D_{2,3}$ and $D_{3,3}$.

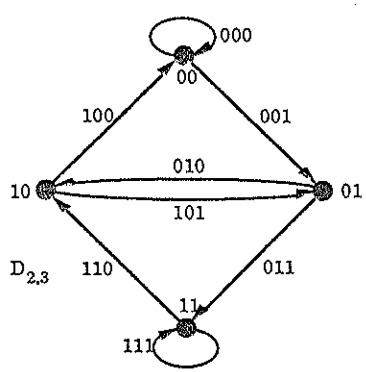

Figure 7.16: The de Bruijn diagram $D_{2,3}$.

We now use the digraphs $D_{n,k}$ to show that, for every pair of positive integers n and k, both greater than 1, there is an (n,k) de Bruijn sequence.

First suppose that the digraph $D_{n,k}$ has a directed Euler tour T say. Now choose the first term of each arc of T in turn to form a sequence σ of n^k terms. We claim that σ is a de Bruijn sequence. To see this, let $w = b_1 b_2 b_3 \ldots b_n$ be any word of length n from Σ. Then this corresponds to the unique arc a from the vertex $b_1 b_2 b_3 \ldots b_{n-1}$ to the vertex $b_2 b_3 b_4 \ldots b_n$. On the tour T the next arc after a must begin $b_2 b_3 b_4 \ldots$, while the next one after that must begin $b_3 b_4 \ldots$, and so on. Thus, in our construction of the sequence σ, we must encounter $w = b_1 b_2 b_3 \ldots b_n$ as a subsequence of n consecutive terms. This shows that each of the n^k words is generated by σ in the desired way and so, since σ has n^k terms, it follows that σ is a de Bruijn sequence, as claimed.

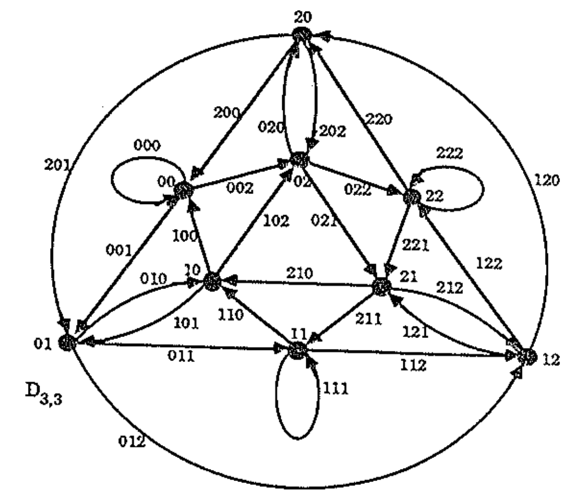

Figure 7.17: The de Bruijn diagram $D_{3,3}$.

We have proved that an (n,k) de Bruijn sequence exists provided the corresponding digraph $D_{n,k}$ has a directed Euler tour. The following theorem guarantees such tours. It was proved by N. G. de Bruijn [15] for the case $k = 2$ and for arbitrary k by I. J. Good [27].

Theorem 7.4 (de Bruijn, 1946; Good, 1946) *For each pair of positive integer n and k, both greater than one, the de Bruijn diagram $D_{n,k}$ has a directed Euler tour.*

Proof By Theorem 7.2 it suffices to show that $D_{n,k}$ is weakly connected and that id $(v) =$ od (v) for each of its vertices v.

Let x and y be two vertices of $D_{n,k}$, say $x = b_1 b_2 b_3 \ldots b_{n-1}$ and $y = c_1 c_2 c_3 \ldots c_{n-1}$. Then there is a directed path from x to y given by the sequence of vertices

$$b_1 b_2 b_3 \ldots b_{n-1},\; b_2 b_3 \ldots b_{n-1} c_1,\; b_3 b_4 \ldots b_{n-1} c_1 c_2,\; \ldots\;, c_1 c_2 c_3 \ldots c_{n-1}.$$

This shows that $D_{n,k}$ is weakly connected, in fact, strongly connected.

Consider again our vertex $x = b_1 b_2 \ldots b_{n-1}$. By the definition of the arcs in $D_{n,k}$, any arc a having x as its tail is of the form $y = b_1 b_2 b_3 \ldots b_{n-1} c_1$. It follows that x has outdegree n. Similarly, any arc b having x as its head is of the form $y = a_1 b_1 b_2 b_3 \ldots b_{n-2}$ and so x has also indegree n. Thus, since $od(x) = id(x)$ for all vertices x, $D_{n,k}$ has a directed Euler tour, as required. □

To illustrate the above discussion, we construct a (3,3) de Bruijn sequence using the de Bruijn diagram of Figure 7.17. As the reader may easily check from the Figure, a directed Euler tour of $D_{3,3}$ is given by the sequence of arcs

200, 000, 001, 011, 111, 112, 122, 222, 220,
202, 022, 221, 212, 121, 211, 110, 101, 010,
100, 002, 021, 210, 102, 020, 201, 012, 120.

Thus, taking the first term of each of these arcs in turn, we get the de Bruijn sequence

2 0 0 0 1 1 1 2 2 2 0 2 2 1 2 1 1 0 1 0 0 2 1 0 2 0 1.

Similarly, using the de Bruijn diagram of Figure 7.16, we may construct the (2,3) de Bruijn sequence

0 0 0 1 1 1 0 1.

We now discuss an application of $(2, k)$ de Bruijn sequences to a problem in telecommunications. To simplify the discussion we restrict our attention to (2,3) de Bruijn sequences, (which are of length eight). Suppose we have a rotating drum with eight segments round its circumference, some of which can conduct an electric current while the others are insulated. Three electric contacts are placed against the drum so that after any rotation of the drum there are three consecutive drum segments touching these contacts, one segment per contact. Then, for example, if we denote a conducting segment by 1 and an insulated one by 0, we get a binary sequence of length three for each position of the drum. The question arises as to whether we can arrange the segments of the drum so that each position gives a different sequence.

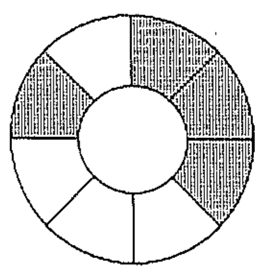

Figure 7.18: A drum design constructed using a (2,3) de Bruijn sequence.

Clearly such an arrangement amounts to finding a (2,3) de Bruijn sequence. Using the one above, we get the drum design shown in Figure 7.18 where the dark segments,

corresponding to the 0's in the sequence, are insulated and the positioning starts at the top and procedes clockwise. Finally we note that this problem occurs in a various forms, ranging from the generation of codes in cryptography to the design of washing machine dials, and is known as the **teleprinter's problem**.

Exercises for Section 7.2

7.2.1 Find $od(v)$ and $id(v)$ for each vertex of the digraph of Figure 7.7.

7.2.2 A digraph D is called k-**regular** if $od(v) = id(v) = k$ for each vertex v of D.

(a) Give an example of a 1-regular digraph with n vertices for each $n \geq 2$.

(b) Give an example of a 2-regular digraph with five vertices.

(c) Prove that given any $n \geq 1$ and any k with $0 \leq k < n$ there is a simple k-regular digraph D with n vertices.

7.2.3 Let D be a digraph with an odd number of vertices. Prove that if each vertex of D has an odd outdegree then there is an odd number of vertices of D with odd indegree.

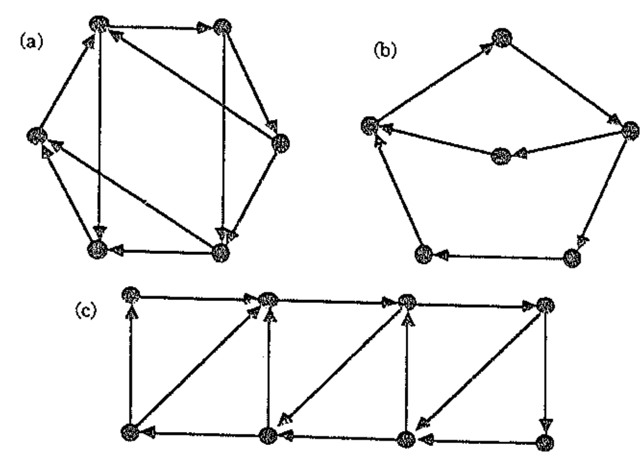

Figure 7.19

7.2.4 Which digraphs of Figure 7.19 are Euler and which have a directed Euler trail?

7.2.5 Let D be an Euler digraph. Prove that D is strongly connected.

7.2.6 Let D be a digraph with a directed Euler trail. Prove that D is unilaterally connected.

7.2.7 Draw the (3,2) de Bruijn diagram and use it to construct a (3,2) de Bruijn sequence.

7.2.8 Draw the (2,4) de Bruijn diagram and use it to construct a (2,4) de Bruijn sequence. Use this sequence to design a rotating drum, having the same properties as that described in the text but with 16 segments round its circumference and four electric contacts.

7.2.9 Prove by induction on n that for each $n \geq 1$ there is a simple digraph D with n vertices $v_1, \ldots, v_n$ such that $od(v_i) = i-1$ and $id(v_i) = n-i$ for each $i = 1, \ldots, n$.

7.2.10 Let D be a digraph such that either every vertex of D has positive outdegree or every vertex of D has positive indegree. Prove that D has a directed cycle.

7.2.11 Let D be a digraph such that $id(v) \geq k$ for every vertex v, where k is some fixed positive integer. Prove that D has a directed cycle of length at least $k+1$.

7.3 Tournaments

A **tournament** is an orientation of a complete graph.

In other words, a tournament is a digraph with no (directed) loops in which any two distinct vertices are joined by exactly one arc.

The number of non-isomorphic tournaments increases sharply with the number of vertices. For example, there is only one tournament with exactly 1 vertex and only one with exactly 2 vertices. There are two tournaments on 3 vertices, four on 4 vertices, 12 on 5 vertices. However there are over 9 million on 10 vertices.

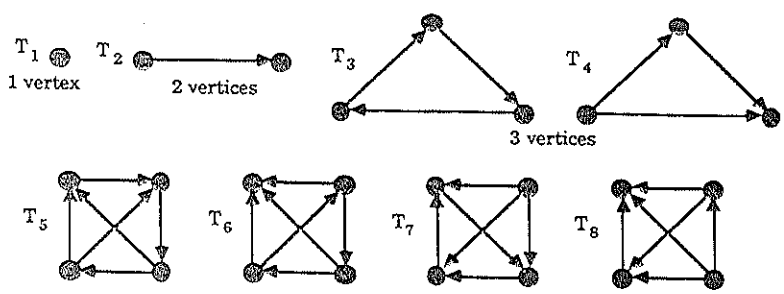

Figure 7.20: The tournaments on at most four vertices.

The tournaments on less than five vertices are shown in Figure 7.20. Here, in the tournaments on four vertices, T_5 is the only one with a directed cycle of length 4, T_6 is the only one with a directed cycle of length 3 and a vertex of indegree 3, T_7 is the only one with a directed cycle of length 3 and a vertex of outdegree 3 and T_8 is the only one with no directed cycle of length 3.

The reason for the name "tournament" is that the digraph can be used to record the results of games in a round-robin tournament in any game in which draws are not allowed, such as tennis. The arc from a to b then indicates that a has beaten b.

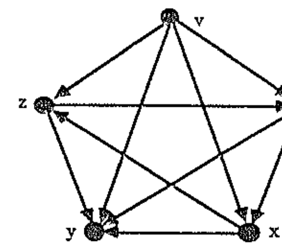

Figure 7.21: A tournament on five vertices.

For example, in the digraph of Figure 7.21 representing a round-robin tournament, v has beaten every other competitor, y has lost each match, each of the other competitors have won two and lost two matches. The vertex v has maximum outdegree 4. Every other vertex can be reached from v by a directed path of length at most 2. This illustrates the following general result:

Theorem 7.5 *Let v be any vertex having maximum outdegree in the tournament T. Then for every vertex w of T there is a directed path from v to w of length at most 2.*

Proof Let $od(v) = m$ and let the vertices joined by an arc *from* v be $v_1, v_2, \ldots, v_m$. If T has n vertices then each of the remaining $n - m - 1$ vertices $u_1, u_2, \ldots, u_{n-m-1}$ are adjacent to v, since T is a tournament, i.e., for these remaining vertices u_j, $1 \leq j \leq n-m-1$, there are arcs *from* u_j *to* v. (See Figure 7.22.)

Then for each i, $1 \leq i \leq m$, the arc from v to v_i gives a directed path of length 1 from v to v_i. It remains to show that there is a directed path of length 2 from v to u_j for each j, $1 \leq j \leq n-m-1$.

Given such a vertex u_j, if there is an arc from v_i to u_j for some i then vv_iu_j gives a directed path of the desired type. However, now suppose there is a u_k, $1 \leq k \leq n-m-1$, such that *no* vertex v_i, $1 \leq i \leq m$, has an arc *from* v_i *to* u_k. Then, because T is a tournament, there must be an arc *from* u_k *to* each of the m vertices v_i. Since we also have an arc *from* u_k *to* v this gives $od(u_k) \geq m+1$. This contradicts the fact that v has maximum outdegree with $od(v) = m$. Thus each u_j must have an arc joining it from some v_i and so the proof is complete by using the directed path vv_iu_j. □

Theorem 7.5 has the following interpretation in round-robin tournaments. Let w be a winner in such a tournament, i.e., any player with the most victories — there may

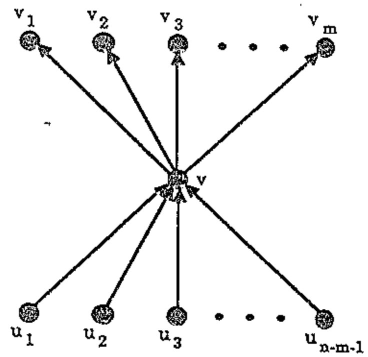

Figure 7.22: Vertex v has maximum outdegree.

be more than one winner. Then w has been defeated only by players who themselves have lost to players defeated by w.

Suppose T is a tournament on n vertices and let v be any vertex of T. Then by $T-v$ we mean the directed graph obtained from T by removing v and all arcs incident with v. Now any two vertices of $T-v$ are joined by exactly one arc, since these two vertices are joined by exactly one arc in T. Hence $T-v$ is also a tournament. We use this property of tournaments in the proof of the next theorem, due to Rédei [53].

A **directed Hamiltonian path** of a digraph D is a directed path in D that includes every vertex of D (once and only once).

Theorem 7.6 (Rédei, 1934) *Every tournament T has a directed Hamiltonian path.*

Proof Assume that T has n vertices. If $n = 1$, 2 or 3 we can easily check from Figure 7.20 that T has a directed Hamiltonian path. Thus we may assume $n \geq 4$.

Fix such an n and assume that the result is true for all tournaments on $n-1$ vertices. Let v be a vertex of T. Then $T-v$ has $n-1$ vertices and so, since by the remarks above $T-v$ is a tournament, there is, by our assumption, a directed Hamiltonian path in $T-v$. Let $P = v_1v_2\dots v_{n-1}$ be such a path.

Now, if there is an arc from v to v_1, then

$$P' = vv_1v_2\dots v_{n-1}$$

is a directed Hamiltonian path in T. Similarly, if there is an arc from v_{n-1} to v then

$$P'' = v_1v_2\dots v_{n-1}v$$

is a directed Hamiltonian path in T. Thus, in both of these cases, we are finished.

Hence we may now suppose that there is no arc from v to v_1 and no arc from v_{n-1} to v. Then there is at least one vertex w on the path P with the property that there is an arc from w to v and w is not v_{n-1} (since v_1 has this property). Let v_i be the

last vertex on P having this property, so that the next vertex v_{i+1} does not have this property. Then, in particular, there is an arc *from* v_i *to* v and an arc *from* v *to* v_{i+1}, as illustrated in Figure 7.23. But then $Q = v_1v_2\dots v_ivv_{i+1}v_{i+2}\dots v_{n-1}$ gives us a directed Hamiltonian path in D. Our proof is now complete by induction. □

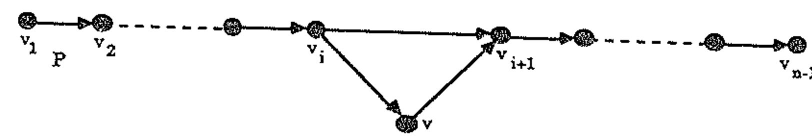

Figure 7.23

A **directed Hamiltonian cycle** in a digraph D is a directed cycle which includes every vertex of D. If D contains such a cycle then D is called **Hamiltonian.**

The previous result shows that every tournament is "nearly Hamiltonian". The next two, due to Camion [11], completely determine when a tournament is Hamiltonian.

Theorem 7.7 *A strongly connected tournament T on n vertices contains directed cycles of length* $3, 4, \dots, n$.

Proof We first show that T contains a directed cycle of length 3. Let v be any vertex of T. Let W denote the set of all vertices w of T for which there is an arc from v to w. Let Z denote the set of all vertices z of T for which there is an arc from z to v. (Note that since T is a tournament $W \cap Z = \emptyset$.)

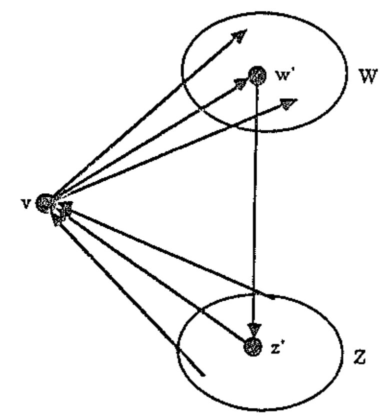

Figure 7.24

Then, since T is strongly connected, W and Z must both be nonempty. (For example, if W were empty, then there would be no arc going out of v, impossible because T is strongly connected). Moreover, again because T is strongly connected, there must be an arc in T going from some w' in W to some z' in Z. This gives the directed cycle $vw'z'v$ of length 3. (See Figure 7.24.)

We now use induction to finish the proof. We suppose that T has a directed cycle C of length k where $k < n$ (and $k \geq 3$) and, using this, we prove that T has a directed cycle of length $k+1$. Let C be given by

$$v_1 v_2 \dots v_k v_1.$$

Suppose that there is a vertex v, not on the cycle C, with the property that there is an arc from v to v_i and an arc from v_j to v for some v_i, v_j on C. Then there must be a vertex v_i on C with an arc from v_{i-1} to v and an arc from v to v_i. Then

$$C' = v_1 v_2 \dots v_{i-1} v v_i v_{i+1} \dots v_k v_1$$

is a directed cycle of length k+1, i.e., of the desired length. (See Figure 7.25.)

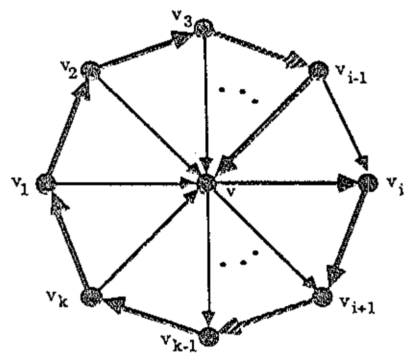

Figure 7.25

If no vertex exists with the above property, then the set of vertices not contained in the cycle can be divided into two distinct sets W and Z, where W is the set of vertices w such that for each i, $1 \leq i \leq k$, there is an arc from v_i to w, and Z is the set of vertices z such that for each i, $1 \leq i \leq k$, there is an arc from z to v_i. If W is empty then the vertices of C and the vertices of Z together make up all the vertices in T. However, by the definition of Z there is no arc from a vertex on C to a vertex in Z, a contradiction since T is strongly connected. Thus W must be nonempty. A similar argument shows that Z is nonempty. Again, since T is strongly connected, there must be an arc from some w' in W to some z' in Z. Then $C' = v_1 w' z' v_3 v_4 \dots v_k v_1$ is a directed cycle of length $k+1$, i.e., of the required length. (See Figure 7.26.) The proof is now complete by induction. □

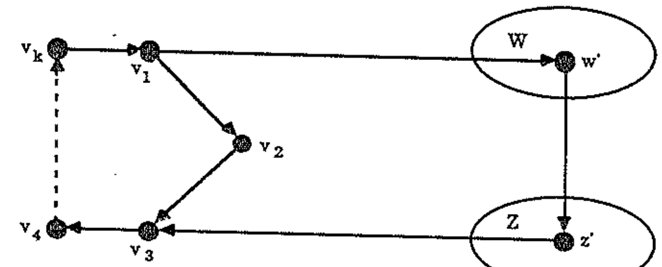

Figure 7.26

Corollary 7.8 (Camion, 1959) *A tournament T is Hamiltonian if and only if it is strongly connected.*

Proof Suppose that T has n vertices. If T is strongly connected then, by the theorem, T must have a directed cycle of length n. Such a cycle is a directed Hamiltonian cycle since it includes every vertex of T. Hence T is Hamiltonian.

Conversely if T is Hamiltonian with directed Hamiltonian cycle $C = v_1 v_2 \dots v_n v_1$, then given any v_i, v_j in the vertex set of T, if $i \geq j$ then $v_j v_{j+1} \dots v_i$ is a directed path P_1 from v_j to v_i while $v_i v_{i+1} \dots v_{n-1} v_n v_1 \dots v_{j-1} v_j$ is a directed path P_2 from v_i to v_j. (See Figure 7.27.) Thus each vertex is reachable from any other vertex and so T is strongly connected, as required. □

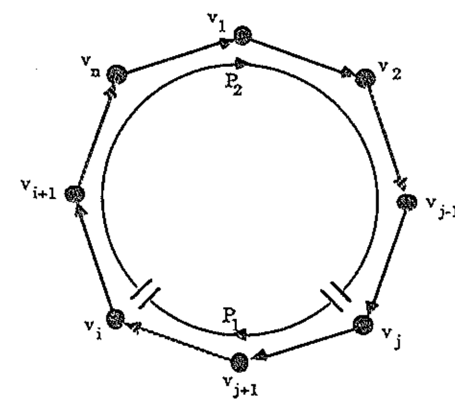

Figure 7.27

We finish this section by noting that strongly connected tournaments may be used to give a ranking of participants in a round-robin competition. In such a competition, there may be several players tied on first place (with the same maximum number of wins). Similarly, there may be several players tied at the next level, and so on. One possible way of ranking the players is to find a directed Hamiltonian path — one exists by Theorem 7.6 — and then rank according to the position on the path. Unfortunately, this system of ranking can be very unfair since in general a tournament can have several directed Hamiltonian paths. However, provided there are at least four players in the competition and the corresponding tournament is strongly connected, there is a procedure that produces a fair ranking of all the players.

Briefly, the procedure is as follows. We first count the number of games won by each player and compare them. This is called the **score** of the player. (See also Exercise 7.3.6.) This gives us an initial ranking where there may be several players on any particular score. In order to distinguish between such players, we next consider the players' **second level scores.** Here a second level score of a player is the sum of the scores of the players she beat. The sequence of second level scores may still result in ties so we compute each player's third level score, i.e., the sum of the second level scores of the players she beat. Even if no ties occur among the second level scores there may appear to be an unfair rearranging of the players' relative positions and so the second level score sequence should be computed regardless.

We now continue the process, finding the fourth level scores, then the fifth level scores, and so on. But when do we stop? The answer to this question lies in matrix theory. Using convergence of matrices one can show (but we won't) that the nth level scores settle down to a fixed pattern after taking a high enough level provided the tournament in question is *strongly connected and has at least four vertices.* For further details and a worked example, we refer the reader to Section 10.7 of Bondy and Murty [7].

Exercises for Section 7.3

7.3.1 In a digraph D, if there is a directed path from the vertex u to the vertex v, then the **distance** from u to v, denoted by $d(u,v)$, is defined to be the length of the shortest such path.

(a) Prove that if u, v and w are vertices in the digraph D such that w is reachable from v and v is reachable from u then w is reachable from u and $d(u,w) \leq d(u,v) + d(v,w)$.

(b) Prove that if u and v are distinct vertices of a tournament T then $d(u,v) \neq d(v,u)$. What are the possible values of $d(u,v)$?

7.3.2 Let T be a tournament on at least two vertices and let U be a proper subset of the vertex set of T. Let $T - U$ denote the digraph obtained from D by deleting all vertices in U and all arcs having either an initial vertex or a terminal vertex in U. Show that $T - U$ is a tournament.

7.3.3 (a) Prove that if five teams play in a round robin tournament then it is possible that all five teams tie for first place, i.e., all have the same number of wins (and losses).

(b) Prove that if six teams play in a round robin tournament then it is *not* possible that all six teams tie for first place.

(c) Prove that if n teams play in a round robin tournament (where $n \geq 3$) then it is possible that all teams tie for first place if and only if n is odd.

7.3.4 Let T be any tournament. Prove that $\overleftrightarrow{T}$, the converse of T, and $\overline{T}$, the complement of T, are isomorphic. (See Exercises 7.1.7 and 7.1.10 for definitions.)

7.3.5 A simple digraph D is called **transitive** if, whenever there is an arc in D from vertex u to vertex v and there is an arc from v to vertex w, then there is an arc from u to w.

(a) Prove that a tournament T is transitive if and only if it has a unique directed Hamiltonian path.

(b) Give an example of a tournament T on four vertices which is not transitive. Justify your example by showing that the transitive condition is not satisfied and also by showing that T has more than one directed Hamiltonian path.

(c) Give an example of a tournament T on four vertices which *is* transitive. What is the unique directed Hamiltonian path in your example T?

(d) Prove that if a simple digraph D has a directed cycle of length three then it is not transitive. Is the converse to this true?

(e) Prove that a tournament T is transitive if and only if it has no directed cycles.

7.3.6 The **score** of a vertex v in a tournament T is defined to be its outdegree. (If T represents a round robin tournament and v a player in this tournament then v's score is the number of games v has won.) If T has vertex set $\{v_1, v_2, \ldots, v_n\}$ where $od(v_1) \leq od(v_2) \leq \cdots od(v_n)$ then the sequence $(od(v_1), od(v_2), \ldots, od(v_n))$ is called a **score sequence** of T.

(a) Find score sequences of the tournaments on four vertices in Figure 7.20 and the tournament on five vertices of Figure 7.21.

(b) Prove that if $(s_1, \ldots, s_n)$ is a score sequence of a tournament T then $\sum_{i=1}^{n} s_i = n(n-1)/2$.

(c) Draw a tournament with score sequence $(0,1,2,3,4)$.

(d) Is it possible for a tournament to have $(3,3,3,3,3)$ as its score sequence?

(e) Prove that a tournament T on n vertices is transitive if and only if it has score sequence $(0,1,2,\ldots,n-1)$.

7.4 Traffic Flow

One-way street assignment is often used by cities to help alleviate traffic flow problems. Given a street map of a city one may ask whether or not it is possible to make each street on the map a one-way street in such a way that one can still drive from many part of the city to any other part (obeying the one-way rules of course!). We may rephrase this question using a graph and an associated digraph as follows.

First construct a graph G in which each vertex represents a street intersection. Join two vertices x and y of G by an edge if it is possible to travel between x and y without passing through any other intersection. (If it is possible to make such a trip between x and y in several ways then there should be an edge corresponding to each of these ways.) The resulting graph G gives us, in effect, a street map of the city, albeit without the street names and possibly not to scale.

If we now want to make each of the streets a one-way street the this amounts to assigning a direction to each of the edges of G, i.e., **orienting** each edge of G. Doing this creates a digraph D which is an orientation of G. Our initial question now amounts to whether or not we can find such a digraph D in which every vertex x is reachable from every other vertex y.

A graph G is called **orientable** if it has a strongly connected orientation.

Clearly then our question asks if our graph G is orientable. To determine when a graph is orientable we first consider the graph G of Figure 7.28.

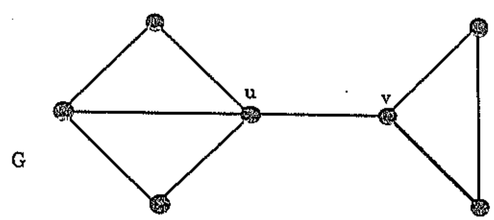

Figure 7.28

In any orientation of G the edge uv is oriented either from u to v or from v to u. If the former then there will be no $v-u$ directed walk in the orientation while if the latter happens then, similarly, v will not be reachable from u. Thus in any event the orientation can not be strongly connected and so G is not orientable.

Of course the reason we are denied a strongly connected orientation here is because the edge uv is a bridge in G. It turns out that bridges are the key to orientability, as the following result, due to Robbins [54], shows.

Theorem 7.9 (Robbins, 1939) *A graph G is orientable if and only if it is connected and has no bridges.*

Proof Clearly if G is not connected then it has no strongly connected orientation. Moreover, if G has a bridge $e = uv$, then, as the argument above shows, in any orientation of G either u is not reachable from v or vice versa. This shows that if G is orientable then G is connected and has no bridges.

Conversely, suppose that G is connected and has no bridges. Any subgraph induced by a single vertex v of G is orientable, since if there are any loops incident with v then changing them to arcs produces a strongly connected digraph with v as the only vertex. Since this shows that G does have vertex induced orientable subgraphs we may now choose a largest possible subset U of the vertex set V of G such that the induced subgraph $H = G[U]$ is orientable. If $H = G$ then, of course, G is orientable and our goal is achieved.

Thus we are left to deal with the case when $H \neq G$, i.e., $U \neq V$. In this case we choose $u \in U, v \notin U$, and, since H is orientable, we may orient the edges of H to get a strongly connected digraph D. Since G is connected and has no bridges it follows from Exercise 7.4.3 (see also Theorem 8.7 of the next chapter) that there are two edge-disjoint paths from u to v in G say

$$P = u_0u_1 \dots u_{m-1}v \text{ and } P' = v_0v_1 \dots v_{n-1}v,$$

where $u_0 = v_0 = u$. Since $u \in H$ but $V \notin H$, there is a vertex $u_i \in P$ which is the last vertex in P belonging to H and there is a vertex $v_j \in P'$ which is the first vertex in P' belonging to H. Now let Q and Q' be those parts of P and P' which start at u_i and v_j respectively, i.e.,

$$Q = u_iu_{i+1} \dots u_{m-1}v \text{ and } Q' = v_jv_{j+1} \dots v_{n-1}v.$$

Orient the edges in Q from u_i to u_{i+1}, from $ui+1$ to u_{i+2}, and so on and orient the edges in Q' from v to v_{n-1}, from v_{n-1} to v_{n-2} and so on (so that the edges of Q' are oriented in reverse to the direction of the path). We illustrate this in Figure 7.29.

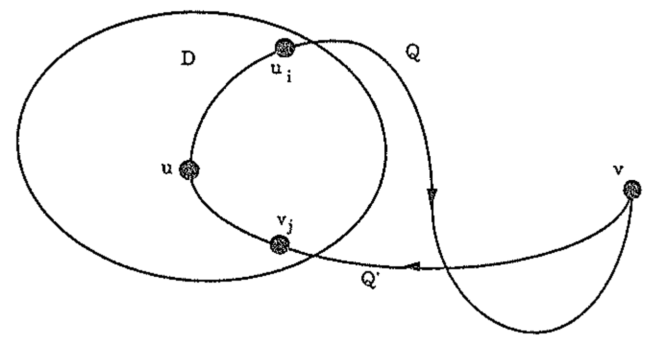

Figure 7.29

Now let D' be the digraph with vertex set U together with all the vertices of Q and Q' and with the arc set of D together with those arcs we have just defined on Q and Q'. Then, since D is strongly connected and the directed walk Q concatenates with the reverse of Q' to give a directed walk from u_i to v_j it follows easily that D' is strongly connected.

However, D' has at least one more vertex than D has, namely v, and so this contradicts our choice of U as a largest possible subset of vertices of G which induces an orientable subgraph. This contradiction shows that $H = G$ after all. □

We now present an algorithm due to J. E. Hopcroft and R. Tarjan [35] which produces a strongly connected orientation for a connected bridgeless graph. The algorithm is not based on the proof above and we leave its justification to Exercise 7.4.5. It uses a labelling technique.

The Hopcroft and Tarjan algorithm

Step 1. Let G be a connected graph with no bridges.

Let x be an arbitrary vertex of G and label x by setting $\lambda(x) = 1$.

Set $L = \{x\}$ and $U = V(G) - \{x\}$. (Here L denotes the set of labelled vertices while U denotesthe set of unlabelled vertices.)

Set $A = \emptyset$. (A denotes the set of arcs produced by orienting edges of G.)

Step 2. Let v be a vertex in L, of highest possible label value, which is adjacent to some vertex $u \in U$. Set $\lambda(u) = \lambda(v) + 1$.

Replace L by $L \cup \{u\}$ and U by $U - \{u\}$ (since u has just been labelled).

Orient the edge vu from v to u and replace A by $A \cup \{(v,u)\}$ (since there is now a new arc from v to u). (Note that, in this process, each new arc of A goes from a labelled vertex to one with a higher label.)

Step 3. If $L \neq V(G)$, repeat Step 2.

Step 4. (When we reach this step we must have $L = V(G)$, i.e., every vertex of G has been labelled. Moreover the arc set A gives an underlying spanning tree of G and one can show that those edges of G not yet oriented join vertices having different label values.)

For each edge xy of G not yet oriented, if $\lambda(x) > \lambda(y)$ then orient xy from x to y. (Since, as noted, the labels of x and y are different, this will always result in an orienting of the edge xy.)

This completes the orientation of G.

We now illustrate the algorithm with the connected bridgeless graph G of Figure 7.30.

Step 1. Choose v_1 as the first vertex in the labelling process. Thus we set $\lambda(v_1) = 1, L = \{v_1\}, U = V(G) - \{v_1\}$ and $A = \emptyset$.

Step 2. Choose $v_2 \in U \cap N(v_1)$. (Another possibility is v_7.) Set $\lambda(v_2) = \lambda(v_1) + 1 = 2$.

L becomes $\{v_1, v_2\}$ and U becomes $\{v_3, \ldots, v_8\}$.

Orient the edge v_1v_2 from v_1 to v_2 so that A becomes $\{(v_1, v_2)\}$.

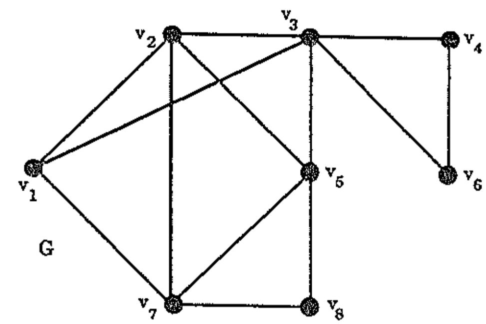

Figure 7.30: A connected graph with no bridges.

Step 3. $L \neq V(G)$ so we return to step 2. (For the sake of brevity, we will omit all but the last of the subsequent step 3's.)

Step 2. Choose $v_3 \in U \cap N(v_2)$. (Other possible choices are v_5 and v_7.) Set $\lambda(v_3) = \lambda(v_2) + 1 = 3$.

L becomes $\{v_1, v_2, v_3\}$ and U becomes $\{v_4, \ldots, v_8\}$.

Orient the edge v_2v_3 from v_2 to v_3 so that A becomes $\{(v_1, v_2), (v_2, v_3)\}$.

Step 2. Choose $v_4 \in U \cap N(v_3)$. (Other possible choices are v_5 and v_6.) Set $\lambda(v_4) = \lambda(v_3) + 1 = 4$.

L becomes $\{v_1, \ldots, v_4\}$ and U becomes $\{v_5, \ldots, v_8\}$.

Orient the edge v_3v_4 from v_3 to v_4 so that A becomes $\{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$.

Step 2. Choose $v_6 \in U \cap N(v_4)$. (There is no other choice.) Set $\lambda(v_6) = \lambda(v_4) + 1 = 5$.

L becomes $\{v_1, \ldots, v_4, v_6\}$ and U becomes $\{v_5, v_7, v_8\}$.

Orient the edge v_4v_6 from v_4 to v_6 so that A becomes

$$\{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_6)\}.$$

Step 2. Now v_3 is the highest labelled vertex with an unlabelled neighbour. Choose $v_5 \in U \cap N(v_3)$. (There is no other choice.) Set $\lambda(v_5) = \lambda(v_3) + 1 = 4$.

L becomes $\{v_1, \ldots, v_6\}$ and U becomes $\{v_7, v_8\}$.

Orient the edge v_3v_5 from v_3 to v_5 so that A becomes

$$\{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_6), (v_3, v_5)\}.$$

Step 2. Now v_5 is the highest labelled vertex with an unlabelled neighbour. Choose $v_7 \in U \cap N(v_5)$. (Another possible choice is v_8.) Set $\lambda(v_7) = \lambda(v_5) + 1 = 5$.

L becomes $\{v_1, \ldots, v_7\}$ and U becomes $\{v_8\}$.

Orient the edge v_5v_7 from v_5 to v_7 so that A becomes

$$\{(v_1,v_2),(v_2,v_3),(v_3,v_4),(v_4,v_6),(v_3,v_5),(v_5,v_7)\}.$$

Step 2. Choose $v_8 \in U \cap N(v_7)$. (There is no other choice.) Set $\lambda(v_8) = \lambda(v_7)+1 = 5$. L becomes $V(G)$ and U becomes the empty set.

Orient the edge v_7v_8 from v_7 to v_8 so that A becomes

$$\{(v_1,v_2),(v_2,v_3),(v_3,v_4),(v_4,v_6),(v_3,v_5),(v_5,v_7),(v_7,v_8)\}.$$

At this stage our (incomplete) orientation of G is as shown in Figure 7.31, together with the labels assigned to the vertices of G.

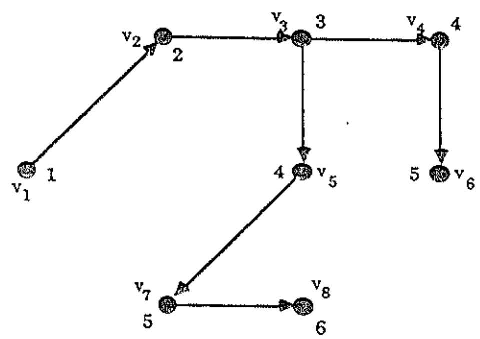

Figure 7.31: A partial orientation of the graph G of Figure 7.28.

Step 3. We now have $L = V(G)$.

Step 4. The edges of G not yet oriented are $v_1v_3, v_1v_7, v_2v_5, v_2v_7, v_3v_6$ and v_5v_8. These are each oriented from the vertex with the higher label to the vertex with the lower label.

This completes the orientation of G, giving the strongly connected digraph shown in Figure 7.32.

Exercises for Section 7.4

7.4.1 Let G be a Hamiltonian graph. Prove that G is orientable without using Theorem 7. 9.

7.4.2 Give a simple construction of a strongly connected orientation for each complete graph with at least three vertices and also for each complete bipartite graph which is not a star graph.

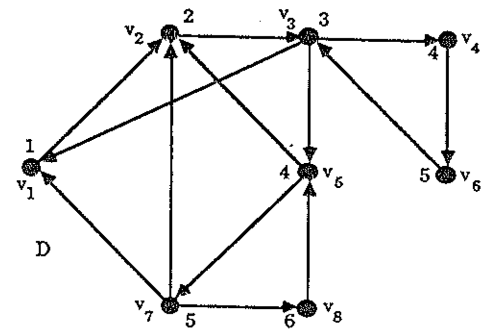

Figure 7.32: A strongly connected orientation of G.

7.4.3 Prove that if G is a connected graph with no bridges then between any two distinct vertices u and v of G there are two edge-disjoint paths.

7.4.4 Using the Hopcroft and Tarjan algorithm, find a strongly connected orientation for the graphs of Figure 7.33.

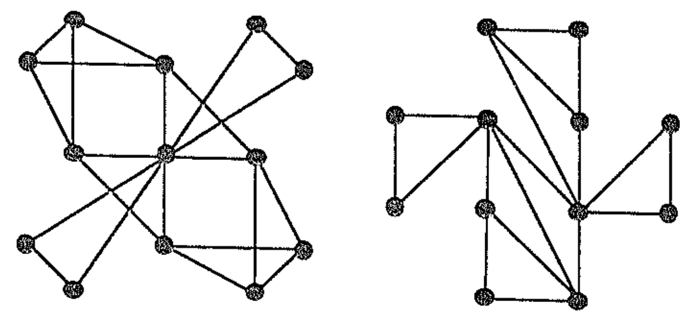

Figure 7.33: Orientable graphs.

7.4.5 This exercise sketches a proof that the Hopcroft and Tarjan algorithm does produce a strongly connected orientation for the connected bridgeless graph G.

(a) Prove that, once all the vertices of G have been labelled in the algorithm, the arc set A gives an underlying spanning tree T of G.

(b) Prove that no arc in the set A joins two vertices having the same label value.

(c) We will say that a vertex a is an **ancestor** of a vertex b in the digraph D finally produced by the algorithm if there is a directed path from a to b involving only arcs from the set A. Prove that x, the first vertex to be labelled, is an ancestor of every vertex.

(d) Let u be a vertex in D different from x and let v be the unique vertex in D such that (v, u) is an arc in A. Let T' be the subtree of T induced by all directed paths in T having u as their initial vertex, i.e., induced by u and all those vertices having u as an ancestor. Using the fact that the edge vu is not a bridge in G, show that there must be an edge in G joining some vertex w in T' to a vertex y not in T'. Prove also that this edge wy is oriented from w to y in D and that y is an ancestor of u.

(e) Prove that if u is a vertex in D different from x then it has a reachable ancestor.

(f) Prove that x is reachable from every other vertex of D and hence that D is strongly connected.

Chapter 8

Networks

8.1 Flows and Cuts

A manufacturer in New Zealand wants to export several boxes of one of his products, clockwork kiwifruit, to a department store in Taiwan. There are various channels through which the boxes can be sent and the digraph of Figure 8.1 represents these, with vertex s as the manufacturer and t the department store. The numbers assigned to each arc represent the maximum loads which each of the corresponding channels can handle.

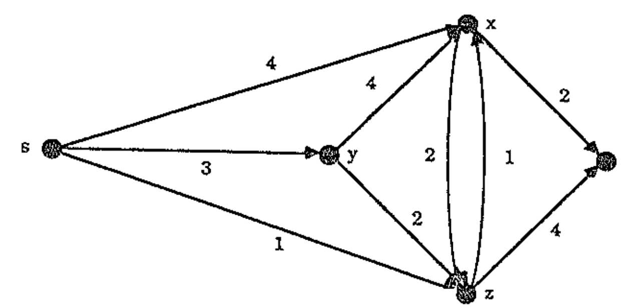

Figure 8.1: A clockwork kiwifruit export network.

The manufacturer wishes to find the maximum number of boxes he can send through the network of channels and "middle men" x, y and z to t so that he never exceeds the permitted capacity of any channel.

Using this example as motivation we now define the concept of a network.

A **network** N is a weakly connected simple digraph in which every arc a of N has been assigned a non-negative integer $c(a)$, called the **capacity** of a.
A vertex s of a network N is called a **source** if it has indegree 0 while a vertex t of N is called a **sink** if it has outdegree 0. Any other vertex of N is called an **intermediate** vertex.