8.3 SEPARATING SETS

LET $u \neq v$ BE VERTICES IN A GRAPH G. A SET $S \subseteq V(G)$, WITH $S \cap \{u,v\} = \phi$, IS CALLED <u>u-v SEPARATING</u> IF $G - S$ IS DISCONNECTED WITH u, v IN DIFFERENT COMPONENTS.

ALSO $F \subseteq E(G)$ IS CALLED <u>u-v SEPARATING</u> IF $G - F$ IS DISCONNECTED WITH u, v IN DIFFERENT COMPONENTS.

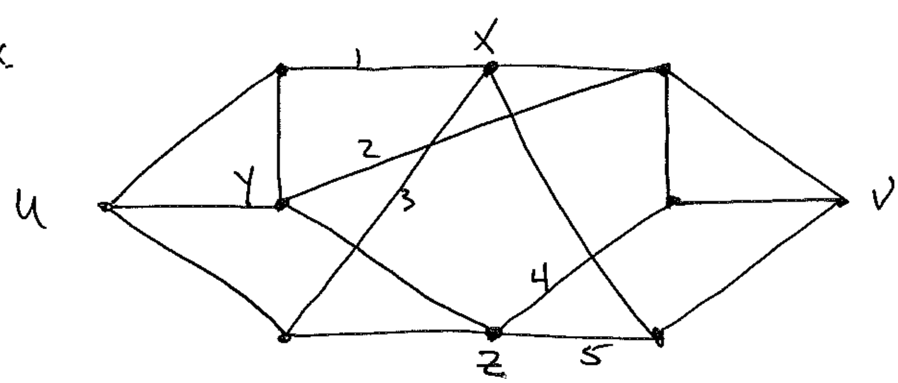
<u>THEOREM</u>

LET $u \neq v$ BE VERTICES IN G.

(1) $S \subseteq V(G)$ IS u-v SEPARATING IF AND ONLY IF EVERY u-v PATH HAS AT LEAST ONE (INTERNAL) VERTEX IN COMMON WITH S

(2) $F \subseteq E(G)$ IS u-v SEPARATING IF AND ONLY IF EVERY u-v PATH HAS AT LEAST ONE EDGE IN COMMON WITH F.

<u>EX</u>



$S = \{x, y, z\}$ is a u-v separating set and $F = \{1, 2, 3, 4, 5\}$ is a u-v separating set of edges. Note F is actually a cut-set.

Verify (1) and (2) for this example.

Proof.

(1) Let $S \subseteq V$ be u-v separating and let P be a u-v path. If P has no (internal) vertex in common with S, then $G-S$ contains P, whence u & v lie in the same component of $G-S$, a contradiction. $\therefore$ S and P must have some vertex in common.

Now suppose every u-v path has an (internal) vertex in common with S. Then $G-S$ can contain no u-v path, whence S is u-v separating.

(2) Let $F \subseteq E$ be u-v separating, and let P be a u-v path in G. If P has no edge in common with F, then $G-F$ contains P, whence u, v lie in the same component of $G-F$, a contradiction. $\therefore$ F and P have some edge in common.

CONVERSELY, SUPPOSE EVERY U-V PATH HAS AN EDGE IN COMMON WITH F. THEN G−F CAN CONTAIN NO U-V PATH, WHENCE F IS U-V SEPARATING.

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MENGER'S THEOREM HAS TWO FORMS: A VERTEX VERSION AND AN EDGE VERSION. BOTH ARE PROVED BY CONSTRUCTING AN ASSOCIATED NETWORK N TO THE GRAPH G, THEN APPLYING THE MAX-FLOW MIN-CUT THEOREM.

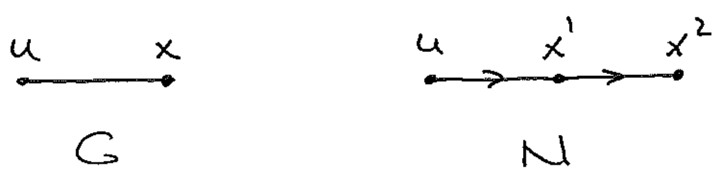
THEOREM (MENGER)
LET U,V BE NON-ADJACENT VERTICES IN G. THEN THE MAXIMUM NUMBER OF INTERNALLY DISJOINT U-V PATHS IN G EQUALS THE MINIMUM NUMBER OF VERTICES IN A U-V SEPARATING SET.

BEFORE PROCEEDING WITH THE PROOF WE FIRST DESCRIBE THE NETWORK N WHICH IS ASSOCIATED TO A GRAPH G AND A PAIR OF VERTICES U, V.

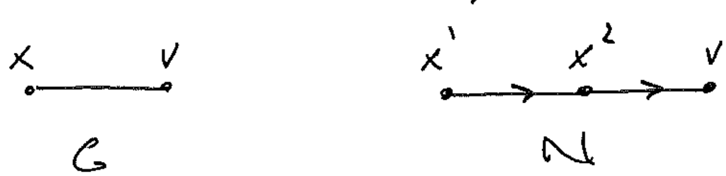
N WILL HAVE SOURCE u AND SINK v.
FOR EACH $x \in V(G)-\{u,v\}$ N WILL CONTAIN TWO VERTICES x^1, x^2 AND A DIRECTED ARC $(x^1,x^2) \in A(N)$

THESE ARCS ARE CALLED <u>INTERNAL ARCS</u>. ALL OTHER ARCS IN N WILL BE CALLED <u>EXTERNAL ARCS</u>.

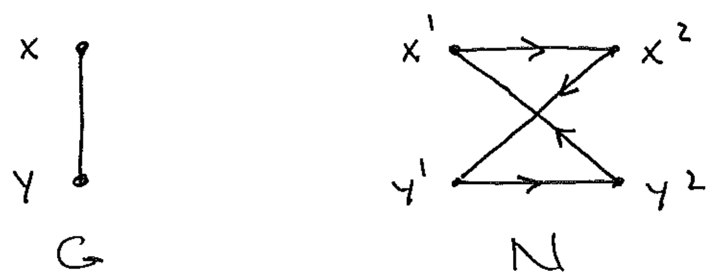
IF x IS ADJACENT TO u IN G THEN N CONTAINS A DIRECTED ARC (u, x^1)



IF x IS ADJACENT TO v IN G THEN N CONTAINS AN ARC (x^2, v)



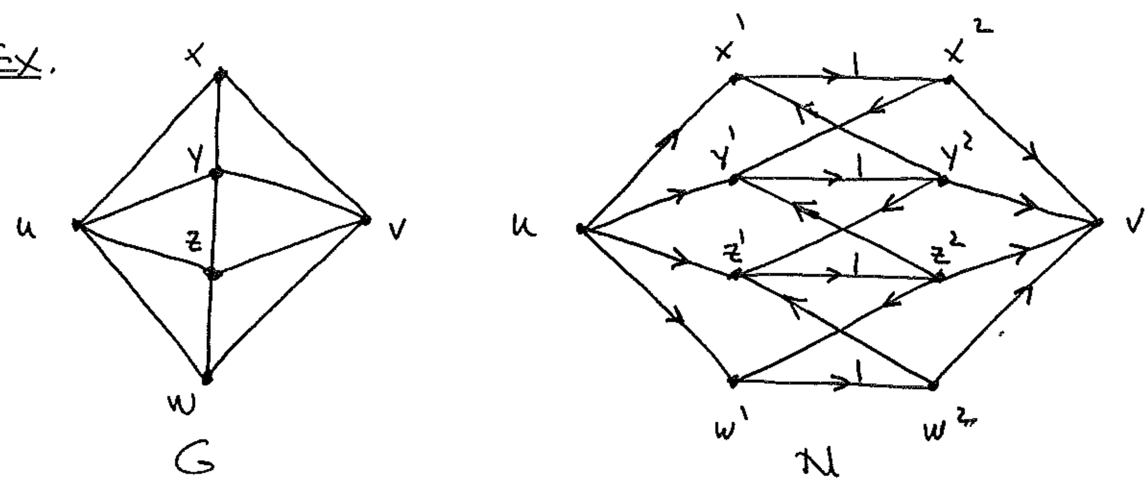
IF $x,y \in V(G)-\{u,v\}$ ARE ADJACENT IN G, THEN N CONTAINS ARCS (x^2,y^1) AND (y^2,x^1)



THESE ARE ALL THE DIRECTED ARCS CONTAINED IN N. DEFINE $c : A(N) \to \mathbb{Z}_+ \cup \{\infty\}$ BY

$$c(a) = \begin{cases} 1 & \text{if } a \text{ is an internal arc} \\ \infty & \text{if } a \text{ is an external arc} \end{cases}.$$

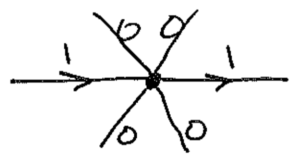
EX.



NOTE THE FOLLOWING.

1.) EACH VERTEX $x^1 \in V(N)$ IS THE ORIGIN OF EXACTLY ONE ARC, NAMELY (x^1, x^2) OF CAPACITY 1.
2.) EACH $x^2 \in V(N)$ IS THE TERMINUS OF JUST ONE ARC, AGAIN (x^1, x^2) OF CAPACITY 1.
3.) IF f IS ANY FLOW IN N, THEN THE FLOW INTO AND OUT OF ANY INTERMEDIATE VERTEX IS EITHER 1 OR 0.
4.) THUS THE FLOW ALONG EACH ARC MUST BE EITHER 0 OR 1.

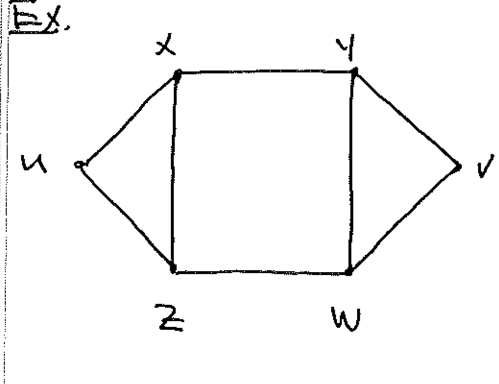
5.) THUS IF THE FLOW THROUGH A VERTEX IS 1, THEN IT HAS EXACTLY ONE INCOMING ARC WITH FLOW 1 AND ONE OUTGOING ARC WITH FLOW 1, ALL OTHER ARCS HAVING FLOW 0.

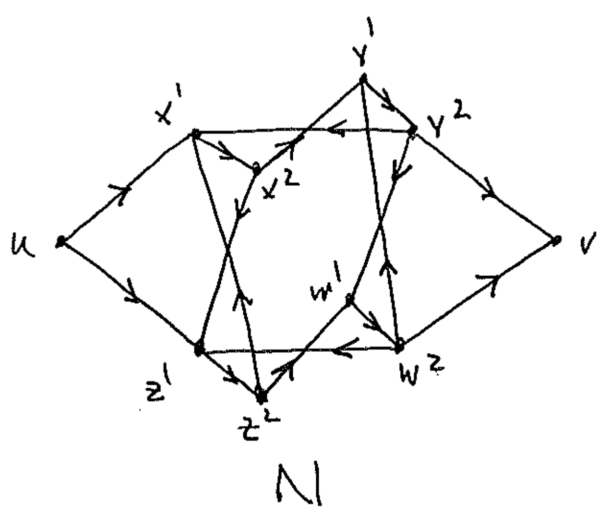


6.) CONSEQUENTLY, IF f IS A FLOW IN N OF VALUE d, THEN N MUST CONTAIN AT LEAST d INTERNALLY DISJOINT DIRECTED u-v PATHS, ALONG WHICH THE FLOW IS 1, ALL OTHER ARCS IN N HAVING FLOW 0.

7.) IN PARTICULAR, IF f IS A MAXIMUM FLOW IN N OF VALUE d, THEN d IS THE MAXIMUM NUMBER OF INTERNALLY DISJOINT DIRECTED u-v PATHS CONTAINED IN N.

EX.





IF THERE WERE A SET OF MORE THAN d INTERNALLY DISJOINT DIRECTED U-V PATHS IN N THEN WE COULD DEFINE A FLOW TO BE 1 ON EACH ARC OF SUCH A PATH, AND 0 ELSEWHERE. THIS FLOW WOULD HAVE VALUE GREATER THAN d.

(NOTE N CONTAINS NO DIRECTED U-V PATH OF INFINITE CAPACITY, OR ZERO CAPACITY.)

8.) THERE IS A ONE-TO ONE CORRESPONDENCE BETWEEN THE SET OF U-V PATHS IN G AND THE SET OF DIRECTED U-V PATHS IN N

A U-V PATH P IN G

$$P : u\, x_1 x_2 \cdots\cdots x_k\, v$$

UNIQUELY DETERMINES A DIRECTED U-V PATH Q IN N

$$Q : u\, x_1^1 x_1^2 x_2^1 x_2^2 \cdots x_k^1 x_k^2\, v \quad .$$

TO SEE THAT THIS MAPPING IS SURJECTIVE LET Q BE ANY DIRECTED U-V PATH IN N

$$Q : u z_1 z_2 \cdots z_\ell v$$

THEN z_1 MUST BE OF THE FORM x_1^1 FOR SOME $x_1 \in V(G)$, AND THEREFORE $z_2 = x_1^2$. SIMILARLY $z_3 = x_2^1$ FOR SOME $x_2 \in V(G)$ ADJACENT TO x_1, AND $z_4 = x_2^2$.

ALSO z_ℓ MUST BE x_j^2 FOR SOME $x_j \in V(G)$ ADJACENT TO v. THUS $\ell = 2j$ IS EVEN AND Q MUST HAVE THE FORM

$$Q : u\, x_1^1 x_1^2 \cdots x_j^1 x_j^2\, v$$

WHICH IS INDUCED BY $P : u x_1 x_2 \cdots x_j v$ IN G.

9.) IT IS CLEAR FROM ABOVE THAT TWO u-v PATHS IN G ARE INTERNALLY DISJOINT IF AND ONLY IF THE CORRESPONDING DIRECTED u-v PATHS IN N ARE INTERNALLY DISJOINT.

10.) THUS IF f IS A MAXIMUM FLOW IN N OF VALUE d, THEN d IS THE MAXIMUM NUMBER OF INTERNALLY DISJOINT u-v PATHS IN G.

PROOF: (OF MENGER'S THEOREM)
LET m BE THE MAXIMUM NUMBER OF INTERNALLY DISJOINT u-v PATHS IN G, AND SAY $\{P_1,..,P_m\}$ A SET OF SUCH PATHS.

LET n BE THE SIZE OF A SMALLEST u-v SEPARATING SET $S \subseteq V(G)$. WE MUST SHOW $m=n$.

BY THE PREVIOUS THEOREM, EACH OF THE PATHS $P_1,..,P_m$ CONTAINS AT LEAST ONE ELEMENT OF S. SINCE THE PATHS $P_1,..,P_m$ ARE (PAIRWISE) INTERNALLY DISJOINT, NO VERTEX IN S LIES IN MORE THAN ONE PATH P_i.

THUS THERE IS AN INJECTION

$$\{P_1,\dots,P_m\} \to S$$

(FOR EACH P_i SELECT A VERTEX v_i IN COMMON WITH S. MAP P_i TO $v_i \in S$.)

IT FOLLOWS THAT $m \leq |S| = n$.

TO SHOW $m \geq n$, WE CONSIDER THE NETWORK N ASSOCIATED TO G DISCUSSED ABOVE. WE HAVE SEEN THAT $m = d$ WHERE d IS THE VALUE OF A MAXIMUM FLOW IN N.

BY THE MAX-FLOW MIN-CUT THEOREM $d = c(X, \bar{X})$ WHERE $A(X, \bar{X})$ IS SOME CUT IN N.

SINCE d IS FINITE, NO ARC IN $A(X,\bar{X})$ CAN HAVE INFINITE CAPACITY, SO $A(X,\bar{X})$ CONTAINS ONLY INTERNAL ARCS, $\therefore d = |A(X,\bar{X})|$.

SINCE $A(X,\bar{X})$ IS A CUT, EACH DIRECTED u-v PATH CONTAINS SOME ARC IN $A(X,\bar{X})$.

DELETING THE ORIGIN OF EACH ARC IN $A(X,\bar{X})$ THUS REMOVES ALL DIRECTED u-v PATHS IN N. IT FOLLOWS THAT

$$\{x \in V(G) : x' \text{ is origin of some } a \in A(X,\bar{X})\}$$

IS A u-v SEPARATING SET IN G. THIS SET CONTAINS $|A(X,\bar{X})| = d$ VERTICES, WHENCE $d \geq n$.

THEREFORE $m \geq n$, AND $m = n$ AS REQUIRED.

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RECALL THE (<u>VERTEX) CONNECTIVITY</u> OF A SIMPLE GRAPH G, DENOTED $\kappa(G)$, IS THE SMALLEST NUMBER OF VERTICES IN G WHOSE DELETION LEAVES A DISCONNECTED GRAPH OR K_1.

G IS CALLED <u>n (VERTEX) CONNECTED</u> IF $\kappa(G) \geq n$ (WHERE $n \geq 1$.) i.e. AT LEAST n VERTICES MUST BE REMOVED FROM G TO PRODUCE A DISCONNECTED GRAPH OR K_1.

<u>THEOREM</u>. (WHITNEY)
A SIMPLE GRAPH G IS n-CONNECTED IF AND ONLY IF FOR ANY PAIR $u \neq v$ OF VERTICES IN G, THERE ARE AT LEAST n INTERNALLY DISJOINT u-v PATHS IN G.

<u>PROOF</u>:
SUPPOSE G IS n-CONNECTED AND LET u, v BE ANY TWO VERTICES IN G. ANY u-v SEPARATING SET MUST THEREFORE HAVE AT LEAST n VERTICES. BY MENGER'S THEOREM G MUST CONTAIN AT LEAST n INTERNALLY DISJOINT u-v PATHS.

CONVERSLEY, SUPPOSE THAT FOR ANY VERTICES $u \neq v$ IN G THERE ARE AT LEAST n INTERNALLY DISJOINT u-v PATHS IN G. BY MENGER'S THEOREM, FOR EACH PAIR $u \neq v$, EVERY u-v SEPARATING SET CONTAINS AT LEAST n VERTICES. THUS AT LEAST n VERTICES MUST BE REMOVED FROM G IN ORDER TO SEPARATE ANY u AND v, i.e. TO CREATE A DISCONNECTED GRAPH. THUS G IS n-CONNECTED.

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MENGERS THEOREM - EDGE VERSION.

LET u, v BE DISTINCT VERTICES IN G. THEN THE MAXIMUM NUMBER OF EDGE DISJOINT u-v PATHS IN G EQUALS THE MINIMUM OF EDGES IN A u-v SEPARATING SET.

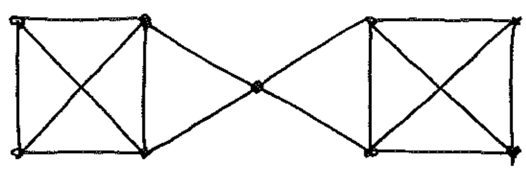
THE PROOF IS SIMILAR TO THAT OF THE NODE VERSION IN THAT IT USES AN AUXILLARY NETWORK N ASSOCIATED TO G. IN THIS CASE THE CONSTRUCTION IS NOT QUITE AS COMPLICATED.

EXCERCISE: READ PROOF P. 285-287

RECALL THE EDGE CONNECTIVITY OF A SIMPLE GRAPH G, DENOTED $\lambda(G)$, WAS DEFINED AS THE SIZE OF A SMALLEST CUT-SET IN G. i.e. $\lambda(G)$ is THE SMALLEST NUMBER OF EDGES WHOSE DELETION RESULTS IN A DISCONNECTED GRAPH OR K_1.

NOTE: BOOK WRITES $K_e(G)$ FOR $\lambda(G)$.

EX. RECALL



$\delta(G) = 3$
$\lambda(G) = 2$
$K(G) = 1$

RECALL ALSO A SIMPLE GRAPH G is CALLED n-EDGE CONNECTED IF $\lambda(G) \geq n$ (WHERE $n \geq 1$.) i.e. AT LEAST n EDGES MUST BE REMOVED IN ORDER TO DISCONNECT G.

THEOREM. (WHITNEY - EDGE VERSION)
A SIMPLE GRAPH G is n-EDGE CONNECTED IF AND ONLY IF FOR ANY PAIR OF DISTINCT $u, v \in V(G)$, THERE EXIST AT LEAST n EDGE DISJOINT u-v PATHS IN G.

PROOF

SUPPOSE G IS n-EDGE CONNECTED, AND LET u, v BE DISTINCT VERTICES IN G. THEN ANY u-v SEPARATING EDGE SET MUST CONTAIN AT LEAST n EDGES. BY MENGER'S THEOREM (EDGE VERSION) G MUST CONTAIN AT LEAST n EDGE-DISJOINT u-v PATHS.

CONVERSELY, SUPPOSE THAT FOR ANY PAIR u, v OF DISTINCT VERTICES, G CONTAINS AT LEAST n-EDGE DISJOINT u-v PATHS. BY MENGER'S THEOREM AGAIN, FOR EVERY PAIR $u \neq v$, EVERY u-v SEPARATING EDGE SET CONTAINS AT LEAST n EDGES. THUS AT LEAST n EDGES MUST BE REMOVED IN ORDER TO SEPARATE ANY u, v; I.E. TO CREATE A DISCONNECTED GRAPH. THUS G IS n-CONNECTED.

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RECALL WE LET $\delta(G)$ DENOTE THE MINIMUM VERTEX DEGREE IN A SIMPLE GRAPH G.

ALSO RECALL:

THEOREM (WHITNEY'S INEQUALITY)
IF G IS A SIMPLE GRAPH THEN

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

PROOF:
LET v BE A VERTEX OF MINIMUM DEGREE $\delta(G)$. THEN DELETING THE $\delta(G)$ EDGES INCIDENT WITH v ISOLATES v, WHENCE $\lambda(G) \leq \delta(G)$.

IF $\lambda(G) = 0$ THEN G IS EITHER DISCONNECTED OR K_1 SO $\kappa(G) = 0$.

IF $\lambda(G) = 1$ THEN G IS CONNECTED WITH A BRIDGE $e \in E(G)$. IN THIS CASE EITHER G HAS A CUT VERTEX (ONE OF e'S ENDS) OR G IS K_2. IN BOTH CASES $\kappa(G) = 1$.

ASSUME NOW THAT $\lambda(G) \geq 2$. FOR CONVENIENCE LET $n = \lambda(G)$. THEN G CONTAINS A SET

$$\{e_1, \ldots, e_n\} \subseteq E(G)$$

WHOSE REMOVAL DISCONNECTS G, BUT

NO SMALLER EDGE SET WILL DO. LET

$$H = G - \{e_1, \ldots, e_{n-1}\}.$$

THEN H IS A CONNECTED SUBGRAPH OF G CONTAINING A BRIDGE e_n. SAY e_n HAS ENDS u, v.

FOR EACH $i = 1, \ldots, n-1$ CHOOSE AN END VERTEX u_i OF e_i SUCH THAT $u_i \neq u$ AND $u_i \neq v$ (NOTE u_i MAY $= u_j$ FOR SOME $i \neq j$). THEN

$$|\{u_1, \ldots, u_{n-1}\}| \leq n-1$$

LET $H' = G - \{u_1, \ldots, u_{n-1}\}$. IF H' IS DISCONNECTED THEN $\kappa(G) \leq n-1 < n = \lambda(G)$. IF H' IS CONNECTED THEN $H' \subseteq H$ CONTAINS THE BRIDGE e_n. EITHER $H' \cong K_2$ OR H' CONTAINS A CUT VERTEX (NAMELY u OR v).

THUS DELETION OF ONE VERTEX FROM H' RESULTS IN A DISCONNECTED GRAPH OR K_1. $\therefore$ $\kappa(G) \leq n = \lambda(G)$ AS REQUIRED.

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