8.2 FORD-FULKERSON ALGORITHM

LET N BE A NETWORK WITH FLOW f. AN <u>f-UNSATURATED TREE</u> IN N IS A TREE T IN THE UNDERLYING GRAPH OF N SATISFYING THE FOLLOWING.

(i) $s \in V(T)$

(ii) FOR ALL $v \in V(T) - \{s\}$, THE UNIQUE s-v PATH IN T IS f-UNSATURATED.

THE FORD-FULKERSON ALGORITHM PRODUCES A MAXIMAL FLOW IN N BY CONSTRUCTING AN INCREASING SEQENCE OF FLOWS, AS IN THE PREVIOUS PROOF.

IT DOES THIS BY STARTING WITH THE TRIVIAL (i.e. ZERO) FLOW f, THEN GROWING AN f-UNSATURATED TREE TO THE POINT WHERE IT INCLUDES THE SINK t ("BREAKTHROUGH").

IT THEN REVISES THE FLOW f USING THE UNIQUE f-INCREMENTING (s-t) PATH IN THIS TREE.

THE ALGORITHM CONTINUES TO GROW TREES, AND REVISE f, UNTIL NO BREAKTHROUGH IS POSSIBLE, I.E. UNTIL THERE IS NO f-INCREMENTING PATH IN N. AT THIS POINT f IS A MAXIMAL FLOW.

VARIABLES

f : A FLOW IN N

T : THE SET OF ARCS IN AN f-UNSATURATED TREE IN N.

$\lambda(x)$: A LABEL PLACED ON VERTEX x WHICH IS THE INCREMENT OF AN s-x PATH.

L : THE SET OF LABELED VERTICES.

S : THE SET OF VERTICES WHICH HAVE BEEN SCANED.

P : AN f-INCREMENTING PATH IN T.

$F = \{xy \in A(N) : f(xy) < c(xy)\}$ "FORWARD" ARCS

$R = \{xy \in A(N) : f(xy) > 0\}$ "REVERSE" ARCS.

THE INPUT TO FORD-FULKERSON IS A NETWORK $N = (V, A)$ WITH SOURCE s, SINK t, AND CAPACITY $c : A \to \mathbb{Z}_+$.

FORD - FULKERSON

1.) $f \leftarrow 0$ (i.e. f is THE ZERO FLOW)

2.) $T \leftarrow \emptyset$, $\lambda(s) \leftarrow \infty$, $L \leftarrow \{s\}$, $S \leftarrow \emptyset$
$F \leftarrow \{xy : f(xy) < c(xy)\}$
$R \leftarrow \{xy : f(xy) > 0\}$

3.) while $L - S \neq \emptyset$

4.) PICK $x \in L - S$

5.) while $\exists\, y \in \overline{L}$ s.t. $xy \in F$ AND $T \cup \{xy\}$ is A TREE

6.) $\lambda(y) \leftarrow \min\{\lambda(x), c(xy) - f(xy)\}$

7.) $T \leftarrow T \cup \{xy\}$

8.) $L \leftarrow L \cup \{y\}$

9.) End while

10.) while $\exists\, y \in \overline{L}$ s.t. $yx \in R$ AND $T \cup \{yx\}$ is A TREE

11.) $\lambda(y) \leftarrow \min\{\lambda(x), f(yx)\}$

12.) $T \leftarrow T \cup \{yx\}$

13.) $L \leftarrow L \cup \{y\}$

14.) End while

15.) $S \leftarrow S \cup \{x\}$

16.) If $t \in L$ (i.e. "BREAKTHROUGH")

17.) $P \leftarrow$ THE UNIQUE f-INCREMENTING s-t PATH IN T

18.) For All $a \in A(P)$

19.) If $a \in F$: $f(a) \leftarrow f(a) + \lambda(t)$

20.) If $a \in R$: $f(a) \leftarrow f(a) - \lambda(t)$

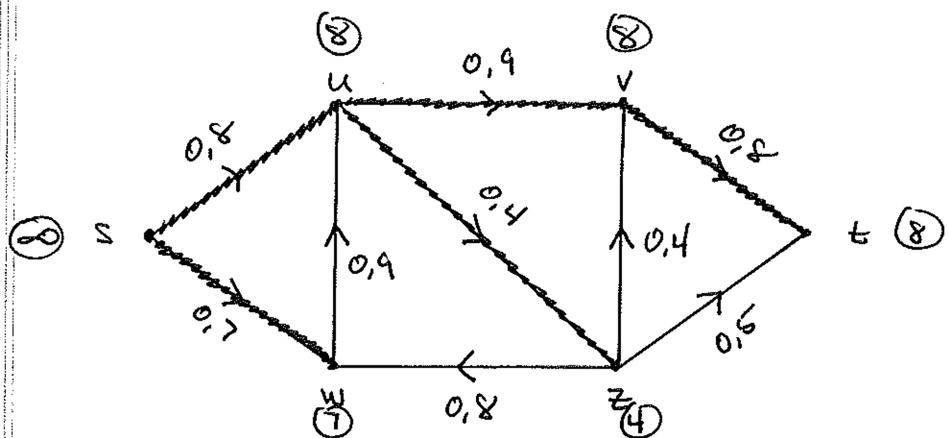
21.) RE-INITIALIZE T, λ, L, S, F, R AS IN (2)

22.) End while

THEOREM

WHEN FORD FULKERSON IS COMPLETE, f IS A MAXIMUM FLOW AND $A(L,\bar{L})$ IS A MINIMUM CAPACITY CUT.

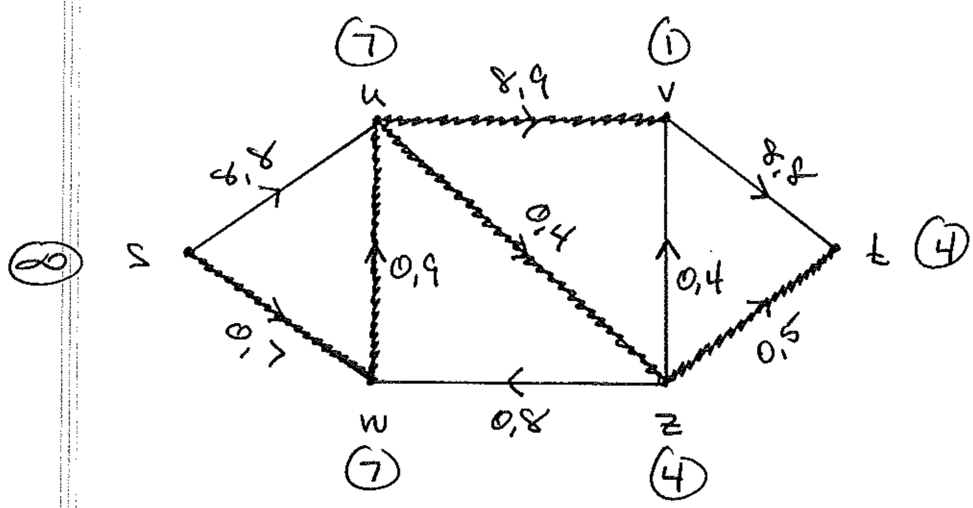
EX. ARCS ARE LABELED f, c.



$L = \{s\}$, $S = \phi$, $F = \{su, sw, wu, uv, uz, zw, zv, vt, zt\}$, $R = \phi$.

$x = s$: $L = \{s, u, w\}$, $S = \{s\}$
$x = u$: $L = \{s, u, w, v, z\}$, $S = \{s, u\}$
$x = w$: $L = \{s, u, w, v, z\}$, $S = \{s, u, w\}$
$x = v$: $L = \{s, u, w, v, z, t\}$, $S = \{s, u, w, v\}$

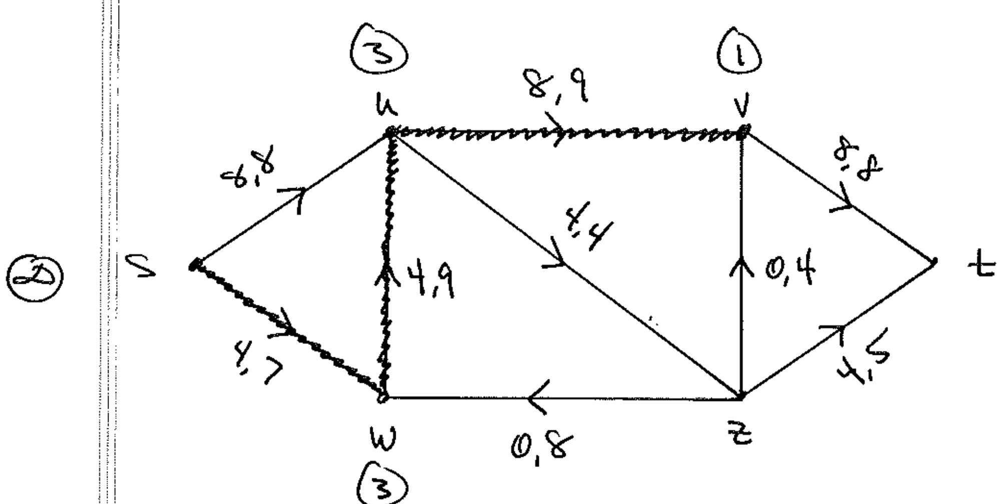
BREAKTHROUGH : REVISE FLOW.
RE-INITIALIZE



$F = \{sw, wu, uv, zw, uz, zv, zt\}$, $R = \{su, uv, vt\}$

$x = s: \quad L = \{s, w\}, \quad S = \{s\}$
$x = w: \quad L = \{s, w, u\}, \quad S = \{s, w\}$
$x = u: \quad L = \{s, w, u, v, z\}, \quad S = \{s, w, u\}$
$x = v: \quad L = \{s, w, u, v, z\}, \quad S = \{s, w, u, v\}$
$x = z: \quad L = \{s, w, u, v, z, t\}, \quad S = \{s, w, u, v, z\}$

BREAKTHROUGH



$F = \{sw, wu, zw, uv, zv, zt\}$
$R = \{su, sw, wu, uz, uv, vt, zt\}$

$x = s: \quad L = \{s, w\}, \quad S = \{s\}$
$x = w: \quad L = \{s, w, u\}, \quad S = \{s, w\}$
$x = u: \quad L = \{s, w, u, v\}, \quad S = \{s, w, u\}$
$x = v: \quad L = \{s, w, u, v\}, \quad S = \{s, w, u, v\}$

NO BREAKTHROUGH POSSIBLE.

$\therefore$ f IS A MAXIMUM FLOW AND $A(\{s,w,u,v\}, \{z,t\})$ IS A MINIMUM CAPACITY CUT.

OBSERVE THAT

$$d = c(\{s,w,u,v\}, \{z,t\}) = 12.$$