8.1 FLOWS AND CUTS

A <u>NETWORK</u> N IS A (WEAKLY) CONNECTED SIMPLE DIGRAPH TOGETHER WITH A NON-NEGATIVE INTEGER VALUED FUNCTION

$$c : A(N) \to \mathbb{Z}_+ = \{0, 1, 2, \dots\}.$$

$c(a)$ IS CALLED THE <u>CAPACITY</u> OF THE ARC $a \in A(N)$.

N IS ALSO SOMETIMES CALLED A <u>CAPACITATED NETWORK</u>.

A VERTEX $s \in V(N)$ IS CALLED A <u>SOURCE</u> IF $id(s) = 0$, AND $t \in V(N)$ IS A <u>SINK</u> IF $od(t) = 0$. ALL OTHER VERTICES ARE CALLED <u>INTERMEDIATE</u>.

FROM NOW ON WE ASSUME OUR NETWORKS CONTAIN EXACTLY ONE SOURCE s, AND EXACTLY ONE SINK t.

WE DENOTE FOR ANY $u \in V(N)$:

$$I(u) = \{\text{incoming arcs } \underline{\text{to}}\ u\}$$

$$O(u) = \{\text{outgoing arcs } \underline{\text{from}}\ u\}.$$

A FLOW (OR FEASIBLE FLOW) IN N IS A FUNCTION

$$f : A(N) \to \mathbb{Z}_+$$

SATISFYING

(i) CAPACITY CONSTRAINT
$f(a) \leq c(a)$ FOR ALL $a \in A(N)$.

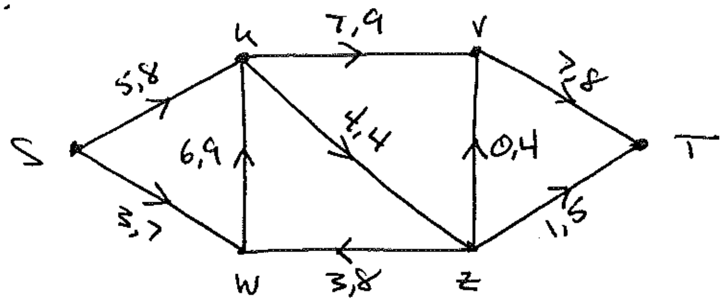
(ii) OUTFLOW FROM s = INFLOW TO t

$$\sum_{a \in O(s)} f(a) = \sum_{a \in I(t)} f(a)$$

(iii) FLOW CONSERVATION
FOR ALL INTERMEDIATE VERTICES x,
OUTFLOW FROM x = INFLOW TO x !

$$\sum_{a \in O(x)} f(a) = \sum_{a \in I(x)} f(a)$$

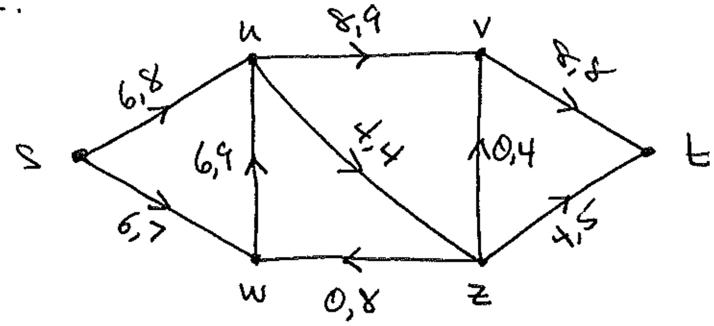
Ex 1.



$d = 8$

EDGES ARE LABELED (f, c).

Ex 2.



$d = 12$

THE NUMBER

$$d = \sum_{a \in O(s)} f(a) = \sum_{a \in I(t)} f(a)$$

is CALLED THE VALUE OF THE FLOW.

GIVEN ANY SUBSETS $X, Y \subseteq V(N)$, DENOTE

$$A(X,Y) = \{\text{ARCS WITH ORIGIN IN } X, \text{ TERMINUS IN } Y\}.$$

GIVEN ANY FUNCTION $g : A(N) \to \mathbb{Z}_+$ (SUCH AS A FLOW OR CAPACITY FUNCTION.)

DEFINE

$$g(X,Y) = \sum_{a \in A(X,Y)} g(a)$$

i.e. $g(X,Y)$ is THE SUM OF ALL THE g-VALUES ALONG ARCS FROM X TO Y.

IF $X \subseteq V(N)$ DENOTE $\overline{X} = V(N) - X$.

A <u>CUT</u> IN N IS A SET OF ARCS OF THE FORM $A(X,\overline{X})$ WHERE $s \in X$ AND $t \in \overline{X}$.

<u>EX. 1</u>

$$X = \{s,u,w\}, \quad \overline{X} = \{v,z,t\}$$

THEN

$$A(X,\overline{X}) = \{uv, uz\} \quad \text{is A CUT}$$

$$C(X,\overline{X}) = 9+4 = 13$$
$$f(X,\overline{X}) = 7+4 = 11$$

Also

$$A(\overline{X},X) = \{zw\} \quad \text{(NOT A CUT)}$$

AND

$$C(\overline{X},X) = 8$$
$$f(\overline{X},X) = 3.$$

Ex. 2

$$X = \{s, u, v, w\}, \quad \bar{X} = \{z, t\}$$

THEN

$$A(X, \bar{X}) = \{uz, vt\} \quad \text{IS A CUT}$$

$$c(X, \bar{X}) = 4 + 8 = 12$$
$$f(X, \bar{X}) = 4 + 8 = 12$$

ALSO

$$A(\bar{X}, X) = \{zv, zw\} \quad (\text{NOT A CUT})$$

$$c(\bar{X}, X) = 4 + 8 = 12$$
$$f(\bar{X}, X) = 0 + 0 = 0$$

THEOREM.

SUPPOSE f IS ANY FLOW IN N WITH VALUE d, AND $A(X, \bar{X})$ IS ANY CUT. THEN

$$d = f(X, \bar{X}) - f(\bar{X}, X) \leq c(X, \bar{X})$$

i.e. d IS THE NET FLOW ACROSS ANY CUT, AND d IS AT MOST THE CAPACITY OF THE CUT.

PROOF

SINCE f IS A FLOW WE HAVE

$$f(\{s\}, V) = d \quad \text{AND} \quad f(V, \{s\}) = 0 .$$

FOR ANY INTERMEDIATE VERTEX u

$$f(\{u\},V) = \sum_{a\in O(u)} f(a) = \sum_{a\in I(u)} f(a) = f(V,\{u\}).$$

THUS

$$f(\{u\},V) - f(V,\{u\}) = 0.$$

IF $A(X,\bar{X})$ IS ANY CUT THEN

$$\begin{aligned} f(X,V) - f(V,X) &= \sum_{x\in X} \left(f(\{x\},V) - f(V,\{x\}) \right) \\ &= f(\{s\},V) - f(V,\{s\}) \quad (\text{SINCE } s\in X.) \\ &= d. \end{aligned}$$

BUT

$$f(X,V) = f(X, X\cup\bar{X}) = f(X,X) + f(X,\bar{X})$$

AND

$$f(V,X) = f(X\cup\bar{X}, X) = f(X,X) + f(\bar{X},X).$$

THEREFORE

$$f(X,\bar{X}) - f(\bar{X},X) = f(X,V) - f(V,X) = d,$$

PROVING THE FIRST CLAIM.

SINCE $f(a) \le c(a)$ FOR ALL $a \in A(N)$ WE HAVE $f(X,\bar{X}) \le c(X,\bar{X})$, SO

$$f(X,\bar{X}) - f(\bar{X},X) \le f(X,\bar{X}) \le c(X,\bar{X}),$$

PROVING THE SECOND CLAIM, AND THE PROOF IS COMPLETE.

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CONSEQUENTLY THE VALUE d OF ANY FLOW MUST SATISFY

$$d \le \min\{c(X,\bar{X}) : A(X,\bar{X}) \text{ is a cut}\}.$$

A FLOW f FOR WHICH EQUALITY HOLDS IS CALLED A <u>MAXIMUM (OR MAXIMAL) FLOW.</u>

IN THE PRECEDING EXAMPLE 2, $d = 12 = c(X,\bar{X})$, SO f IS A MAXIMUM FLOW AND $A(X,\bar{X})$ IS A MINIMUM (CAPACITY) CUT.

NOTE IN GENERAL IF N HAS n VERTICES ($\therefore$ $n-2$ INTERMEDIATE VERTICES) THEN N CONTAINS 2^{n-2} DISTINCT CUTS. (ONE FOR EACH SUBSET OF {INTERMEDIATE VERTICES}.)

IN EX. 2 N HAS $2^4 = 16$ CUTS. FORTUNATELY IT IS NOT NECESSARY TO CHECK EACH OF THESE CUTS TO SEE THAT $A(X,\bar{X})$ IS MINIMUM CAPACITY. THIS IS SINCE

$$d \le c(X,\bar{X})$$

HOLDS FOR ANY FLOW AND ANY CUT. THUS IF EQUALITY IS EVER ATTAINED, THE FLOW MUST BE MAXIMAL AND THE CUT MUST BE MINIMUM. I.e. WE'VE PROVED:

<u>LEMMA</u>
IF f IS A FLOW IN N OF VALUE d AND $A(X,\bar{X})$ IS A CUT, AND IF

$$d = c(X,\bar{X}),$$

THEN f IS MAXIMUM AND $A(X,\bar{X})$ IS A MINIMUM (CAPACITY) CUT.

LET $T: v_0 v_1 \cdots v_k$ BE A (NOT NECESSARILY DIRECTED) TRAIL IN THE UNDERLYING GRAPH OF N.

WE CALL AN EDGE $v_{i-1}v_i$ $(1 \le i \le k)$ A <u>FORWARD ARC OF T</u> IF $(v_{i-1}, v_i) \in A(N)$, AND A <u>REVERSE ARC OF T</u> IF $(v_i, v_{i-1}) \in A(N)$.

LET f BE A FLOW IN N. WE ASSOCIATE TO EACH ARC a IN T A QUANTITY $i(a)$ CALLED ITS <u>INCREMENT</u>:

$$i(a) = \begin{cases} c(a) - f(a) & \text{IF } a \text{ IS FORWARD IN } T \\ f(a) & \text{IF } a \text{ IS REVERSE IN } T \end{cases}$$

WE WRITE $i(a) = i_T(a)$ WHEN WE WISH TO EMPHASIZE THE DEPENDENCE ON T.

$i(a)$ IS THE AMMOUNT BY WHICH f COULD BE INCREASED ON A FORWARD ARC AND STILL OBEY THE CAPACITY CONSTRAINT $f(a) \le c(a)$; AND THE AMMOUNT BY WHICH f COULD BE DECREASED ON A REVERSE ARC AND STILL SATISFY $f(a) \ge 0$.

WE DEFINE THE <u>INCREMENT OF f ALONG T</u> TO BE THE QUANTITY

$$i(T) = \min\{ i(a) : a \text{ IS AN ARC IN } T \}.$$

WE SOMETIMES WRITE $i_f(T) = i(T)$ TO EMPHASIZE THE DEPENDENCE OF THIS QUANTITY ON f.

WE SAY THE TRAIL T IS f-SATURATED IF $i(T) = 0$, AND f-UNSATURATED IF $i(T) > 0$. AN f-INCREMENTING TRAIL (ALSO f-AUGMENTING) IS AN f-UNSATURATED TRAIL FROM SOURCE s TO SINK t.

$$T: \; s = v_0 v_1 v_2 \cdots v_k = t$$

LEMMA

LET T BE AN f-INCREMENTING TRAIL IN N. THEN THE FUNCTION $f_1 : A(N) \to \mathbb{Z}$ DEFINED BY

$$f_1(a) = \begin{cases} f(a) + i(T) & \text{IF } a \text{ IS FORWARD IN } T \\ f(a) - i(T) & \text{IF } a \text{ IS REVERSE IN } T \\ f(a) & \text{IF } a \text{ IS NOT IN } T \end{cases}$$

IS A FLOW IN N. IF f HAS VALUE d, THEN f_1 HAS VALUE $d + i_f(T)$ AND T IS f_1-SATURATED.

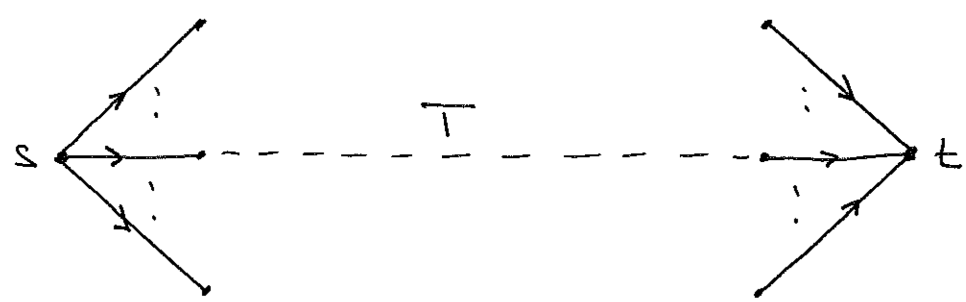
PROOF.

FIRST OBSERVE THAT $0 \le f_1(a) \le c(a)$ FOR ALL $a \in A(N)$ IS OBVIOUS FROM THE DEFINITION OF $i(T)$. THUS $f_1 : A(N) \to \mathbb{Z}_+$ AND THE CAPACITY CONSTRAINT IS SATISFIED!

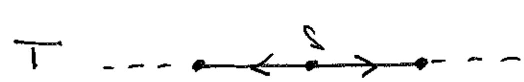
(i) $f_1(a) \le c(a)$ FOR ALL $a \in A(N)$.

IF T IS INCIDENT WITH s JUST ONCE THEN BY THE DEFINITION OF f_1:

$$\sum_{a \in O(s)} f_1(a) = \sum_{a \in O(s)} f(a) + i_f(T)$$



IF T PASSES THROUGH s MORE THAN ONCE, THEN SINCE s HAS ONLY OUTGOING ARCS, THE CONTRIBUTIONS OF $i(T)$ TO $\sum_{O(s)} f_1(a)$ ALONG FORWARD AND REVERSE ARCS CANCEL.



THUS IN THIS CASE ALSO

$$\sum_{a \in O(s)} f_1(a) = \sum_{a \in O(s)} f(a) + i_f(T) .$$

Similar reasoning shows that

$$\sum_{a \in I(t)} f_1(a) = \sum_{a \in I(t)} f(a) + i_f(T) .$$

Since f is a flow we have $\sum_{O(s)} f(a) = \sum_{I(t)} f(a)$, whence

$$\text{(ii)} \quad \sum_{a \in O(s)} f_1(a) = \sum_{a \in I(t)} f_1(a) .$$

Now let x be an intermediate vertex in N. Since f is a flow

$$(*) \quad \sum_{a \in O(x)} f(a) = \sum_{a \in I(x)} f(a) .$$

If x is not incident with T then the flow conservation condition

$$\text{(iii)} \quad \sum_{a \in O(x)} f_1(a) = \sum_{a \in I(x)} f_1(a)$$

follows immediately from the definition of f_1.

If x is incident with T then there are 4 cases to consider:

T 

(1) YX, XZ FORWARD WRT T: $Y \longrightarrow X \longrightarrow Z$

IN THIS CASE $i_f(T)$ IS ADDED TO BOTH LHS AND RHS OF * TO OBTAIN (iii).

(2) YX, XZ REVERSE WRT T: $Y \longleftarrow X \longleftarrow Z$

$i_f(T)$ IS SUBTRACTED FROM BOTH LHS AND RHS OF * TO OBTAIN (iii).

(3.) YX FORWARD, XZ REVERSE: $Y \longrightarrow X \longleftarrow Z$

$i_f(T)$ IS BOTH ADDED AND SUBTRACTED FROM RHS OF * TO OBTAIN (iii).

(4) YX REVERSE, XZ FORWARD: $Y \longleftarrow X \longrightarrow Z$

IN THIS CASE $i_f(T)$ IS ADDED AND SUBTRACTED FROM LHS OF * TO REACH (iii).

IN ALL CASES THE FLOW CONSERVATION CONDITION (iii) HOLDS FOR f_1. WE'VE SHOWN THAT f_1 IS A FLOW.

OUR ARGUMENT FOR (ii) ACTUALLY SHOWED THAT f_1 HAS VALUE $d + i_f(T)$.

THE DEFINITION OF $i_f(T)$ AND f_1 SHOWS THAT THE INCREMENT OF f_1 ALONG SOME ARC OF T MUST BE ZERO, WHENCE $i_{f_1}(T)=0$, i.e. T is f_1-SATURATED.

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THEOREM (MAX FLOW - MIN CUT, FORD - FULKERSON)
LET N BE A NETWORK WITH CAPACITY FUNCTION c. THEN THERE EXISTS A MAXIMUM FLOW IN N. i.e. THERE EXISTS A FLOW WITH VALUE

$$d=\min\{c(X,\bar{X}) : A(X,\bar{X}) \text{ is a cut}\}$$

PROOF:
LET f_0 BE ANY FLOW IN N, SAY OF VALUE d_0. (f_0 CERTAINLY EXISTS. TAKE $f_0=0$ IF NECESSARY.)
WE KNOW THAT

$$d_0 \le \min\{c(X,\bar{X}) : A(X,\bar{X}) \text{ is a cut}\}.$$

DEFINE A SUBSET $X_0 \subseteq V(N)$ BY

$$X_0=\{s\}\cup\{x\in V : \exists \text{ an } f_0\text{-UNSATURATED } s\text{-}x \text{ TRAIL}\}.$$

EITHER $t\in X_0$ OR $t\in \bar{X}_0 = V - X_0$.

SUPPOSE $t \in X_0$. THEN N CONTAINS AN f_0-INCREMENTING TRAIL T. (i.e. T is AN f_0-UNSATURATED s-t TRAIL, WITH $i_{f_0}(T) > 0$.)

BY THE PREVIOUS LEMMA, THE FUNCTION f_1 ON $A(N)$ DEFINED BY

$$f_1(a) = \begin{cases} f_0(a) + i_{f_0}(T) & \text{IF } a \text{ IS FORWARD IN } T \\ f_0(a) - i_{f_0}(T) & \text{IF } a \text{ IS REVERSE IN } T \\ f_0(a) & \text{IF } a \text{ IS NOT IN } T \end{cases}$$

IS A NEW FLOW IN N OF VALUE

$$d_1 = d_0 + i_{f_0}(T) > d_0 .$$

ALSO T IS f_1-SATURATED.

NOW FORM THE SET

$$X_1 = \{s\} \cup \{x \in V : \exists \text{ AN } f_1\text{-UNSATURATED } s\text{-}x \text{ TRAIL}\}$$

ASSOCIATED TO THE REVISED FLOW f_1.

AS BEFORE IF $t \in X_1$ WE CAN CREATE A NEW FLOW f_2 OF VALUE $d_2 > d_1$.

CONTINUING IN THIS MANNER WE CONSTRUCT A SEQUENCE OF FLOWS $\{f_i\}$ OF VALUE $\{d_i\}$ WITH ASSOCIATED SET $\{X_i\}$. IF AT ANY STAGE $t \in X_i$ WE CREATE A REVISED FLOW f_{i+1} OF VALUE $d_{i+1} > d_i$.

THE SEQUENCE OF FLOW VALUES $d_0 < d_1 < d_2 < \dots$ IS BOUNDED ABOVE BY

$$\min\{c(X,\bar{X}) : A(X,\bar{X}) \text{ is a cut}\}.$$

THEREFORE THIS PROCESS MUST TERMINATE WITH A FLOW f HAVING ASSOCIATED SET

$$X = \{s\} \cup \{x \in V : \exists \text{ an } f\text{-unsaturated } s-x \text{ trail}\}$$

SUCH THAT $t \notin X$, i.e. $t \in \bar{X}$.

IN THIS CASE $A(X,\bar{X})$ IS A CUT IN N. WE CLAIM THAT f IS A MAXIMAL FLOW AND THAT $A(X,\bar{X})$ IS A MINIMUM CAPACITY CUT.

LET $x \in X$. THEN N CONTAINS AN f-UNSATURATED $s-x$ TRAIL

$$T: \; s = v_0, v_1, \dots, v_k = x$$

IF $a = (x,y) \in A(X,\overline{X})$ (SO THAT $y \in \overline{X}$) AND $f(a) < c(a)$ THEN T CAN BE EXTENDED TO A TRAIL:

$$T' : \overset{s}{\overset{\parallel}{v_0}}, \ldots, \overset{x}{\overset{\parallel}{v_k}}, y$$

which is still f-UNSATURATED. ($i(a) > 0$ SO THAT $i_f(T') > 0$,) THIS IMPLIES $y \in X$ WHICH CONTRADICTS $y \in \overline{X}$. THEREFORE NO SUCH ARC a EXISTS.

Similarly IF $a = (y,x) \in A(\overline{X}, X)$ AND $f(a) > 0$, THEN T CAN BE EXTENDED TO

$$T'' : \overset{s}{\overset{\parallel}{v_0}}, \ldots\ldots, \overset{x}{\overset{\parallel}{v_k}}, y$$

which is AGAIN f-UNSATURATED. ($i(a) > 0$ SO $i_t(T'') > 0$.) AGAIN $y \in X$ CONTRADICTING $y \in \overline{X}$, SO NO SUCH ARC a EXISTS.

THUS

$$a \in A(X,\overline{X}) \Rightarrow f(a) = c(a)$$

AND

$$a \in A(\overline{X},X) \Rightarrow f(a) = 0 .$$

IT FOLLOWS THAT

$$f(X,\overline{X}) = c(X,\overline{X}) \quad \text{AND} \quad f(\overline{X},X) = 0.$$

BY A PREVIOUS THEOREM THE FLOW VALUE d IS

$$d = f(X,\overline{X}) - f(\overline{X},X) = c(X,\overline{X}).$$

THEREFORE f IS A MAXIMUM FLOW AND $A(X,\overline{X})$ IS A MINIMUM CAPACITY CUT.

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