CMPE 177 8-5-09

Defn

Let D BE A DIGRAPH

$V(D) = \{v_1, \ldots, v_n\}$, $A(D) = \{a_1, \ldots, a_m\}$

THE ADJACENCY MATRIX $A = (a_{ij})$ is $n \times n$ MATRIX WITH:

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in A(D) \\ 0 & \text{otherwise.} \end{cases}$$

WRITE $A(D)$ for ADJ. MATRIX.

NOTE $A(D)$ may not be Symmetric.

THE INCIDENCE MATRIX $M = M(D)$ $= (m_{ij})$ $1 \le i \le n$, $1 \le j \le m$ WITH:

$$m_{ij} = \begin{cases} 0 & \text{if } v_i, a_j \text{ not INCIDENT} \\ -1 & \text{if } v_i \text{ origin of } a_j \\ +1 & \text{if } v_i \text{ Terminus of } a_j \end{cases}$$

(7.2) DEGREES:

Defn THE indegree $id(v)$ of $v \in V(D)$ is # of arcs having v as terminus. THE outdegree $od(v)$ = # ARCS HAVING v AS ORIGIN. THE Degree of v is ITS Degree in UNDERLYING GRAPH of D. SO

$$\deg(v) = id(v) + od(v).$$

THM

Let $V(D) = \{v_1, \ldots, v_n\}$ then

$$\sum_{i=1}^{n} id(v_i) = \sum_{i=1}^{n} od(v_i) = |A(D)|.$$

Proof: Exercise

We have directed analogs of **Euler Trail** & **Euler Tour**.

Eulerian Circuit

A directed trail in D that includes all edges.

Defn D is **Eulerian** iff it contains an Eulerian circuit.

D is <u>semi-Eulerian</u> iff contains a (non-closed) Eulerian trail.

<u>THM</u>

Let D be (weakly) connected with at least one arc. Then D is Eulerian iff for all $v \in V(D)$: $id(v) = od(v)$.

<u>Proof:</u>

($\Rightarrow$) when an Eulerian Circuit passes through $v \in V(D)$, it contributes 1 to the $id(v)$ & 1 to $od(v)$.

Since each arc belongs to this circuit, and since $id(v)$ & $od(v)$

Are the sums of these contributions: $id(v) = od(v)$.

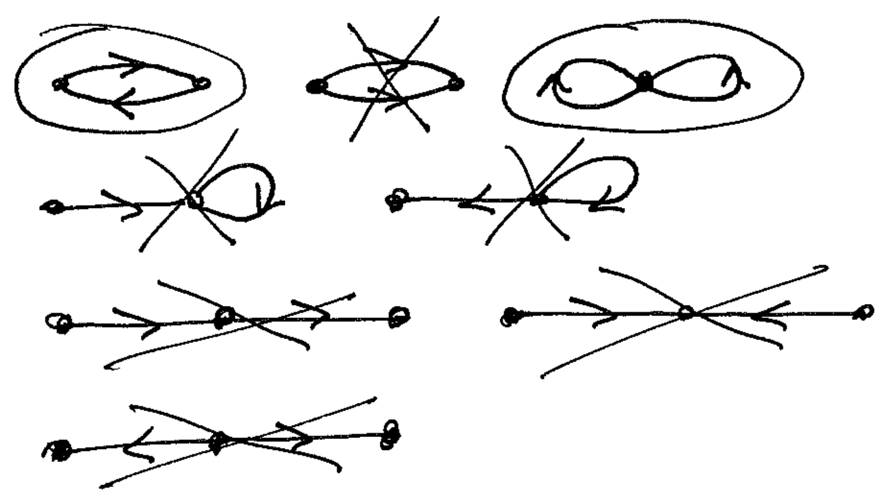
($\Leftarrow$) Suppose $id(v) = od(v)$ for all $v \in V(D)$, must construct an Eulerian circuit in D.

Use induction on $m = |A(D)|$.

Base case:

$m = 1$: ~~$\circ \rightarrow \circ$~~ OR $id(v) = 1$, $od(v) = 1$

$m = 2$:



let $m > 1$ and assume that any digraph with fewer than m arcs which satisfies $id(v) = od(v)$ for all vertices v, contains an Eulerian circuit.

Let $|A(D)| = m$.

<u>Claim</u>: D contains a dir. cycle.

<u>Pf</u>: let T be any dir trail in D. Say u is origin & v terminus of T.

T 

Since $id(v) = od(v) > 0$ there is at least one arc whose origin is v and which does not belong to T.

$\therefore T$ can be extended by one arc. $\therefore T$ can be extended to a DIR. CYCLE. ///

CALL THE DIR CYCLE C,

IF C CONTAINS ALL ARCS, we're DONE. Let $H_1, H_2, \ldots, H_k$ BE THE (weak) COMPONENTS OF $D - E(C)$.

NOTE EACH H_i $(1 \le i \le k)$ HAS equal in & out-degrees. (why?) Also H_i HAS fewer than m arcs. So By Ind. Hyp. H_i CONTAINS AN EULERIAN CIRCUIT C_i

NOW BUILD AN EULERIAN CIRCUIT IN D BY TRAVERSING C UNTIL A NON-ISOLATED VERTEX IN SOME H_i IS ENCOUNTERED. THEN TRAVERSE C_i. THEN CONTINUE ALONG C. Proceed until we return to initial vertex.

///.

<u>THM</u>

Let D BE A (weakly) CONN. DIGRAPH with at least 2 VERTICES, THEN D IS SEMI-EULERIAN IFF THERE EXIST $u, v \in V(D)$ S.T.

$$od(u) = id(u) + 1 \quad \& \quad id(v) = od(v) + 1.$$

and $id(w) = od(w)$ for all other $w \in V(D)$.

(7.3) TOURNAMENTS

Defn

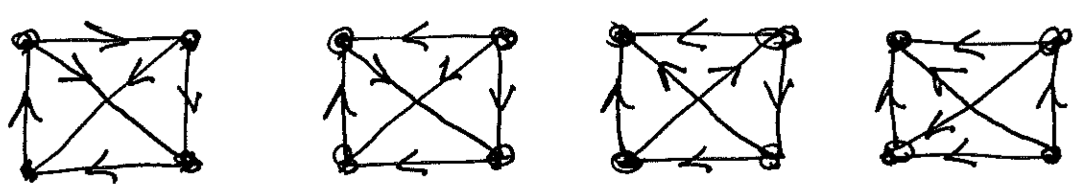
A TOURNAMENT IS AN ORIENTATION OF K_n, i.e. A DIGRAPH WHOSE UNDERLYING GRAPH IS K_n.

IF T IS A TOURNAMENT (with n VERTICES) THEN

$$|A(T)| = \binom{n}{2} = \frac{n(n-1)}{2}$$

$\therefore$ K_n HAS $2^{\binom{n}{2}}$ ORIENTATIONS, ~~i.e.~~

Ex. K_4 HAS $2^{\binom{4}{2}} = 2^6 = 64$ ORIENTATIONS. UP TO ISOMORPHISM, THERE ARE ONLY 4 TOURNAMENTS ON 4 VERTICES:



<u>EXERCISE:</u>

~~THERE ARE~~ $2^{\binom{5}{2}} = 2^{10} = 1024$ ORIENTATIONS OF K_5, BUT ONLY 12 (UP TO DIGRAPH ISOMORPHISM) TOURNAMENTS ON 5 VERTICES. DRAW THEM.

WHY 'TOURNAMENT'?

$$x \longrightarrow y$$

'x beat y'

<u>Defn</u> A DIGRAPH D IS CALLED <u>HAMILTONIAN</u> IFF IT CONTAINS A DIR. CYCLE (called A <u>HAMILTONIAN CYCLE</u>) THAT INCLUDES EVERY VERTEX.

Defn

D is called Semi-Hamiltonian iff it contains a (non-closed) dir. path (called a Hamiltonian path) which includes every vertex.

Thm (Rédei)

Every tournament is either Hamiltonian or Semi-Hamiltonian.

Proof:

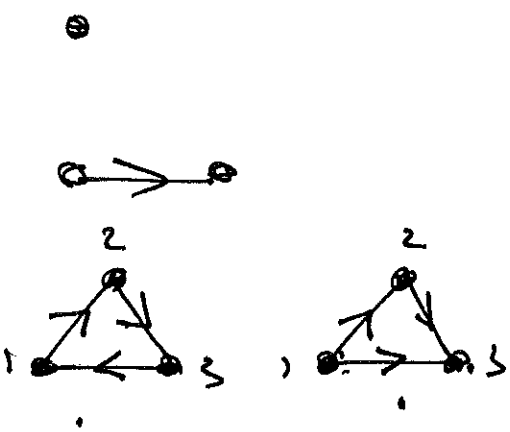
Induction on $n = |V(T)|$.

BASE:

$n = 1$

$n = 2$

$n = 3$



$1 \to 2 \to 3$

INDUCTION

SUPPOSE $n = |V(T)| \geq 3$. ASSUME ANY TOURNAMENT ON $n-1$ VERTICES CONTAINS A DIR. HAMILTONIAN PATH.

PICK ANY $v \in V(T)$. OBSERVE THAT $T - v$ IS A TOURNAMENT ON $n-1$ VERTICES. BY IND. HYP. $T - v$ CONTAINS A HAMILTONIAN PATH.

$P: \; v_1 \to v_2 \to v_3 \to \cdots \to v_{n-1}$

If $(v_{n-1}, v) \in A(T)$ then P can be extended to

$$v_1 \to \cdots \to v_{n-1} \to v$$

which is a Hamiltonian path in T. Likewise if $(v, v_1) \in A(T)$, then P can be extended to

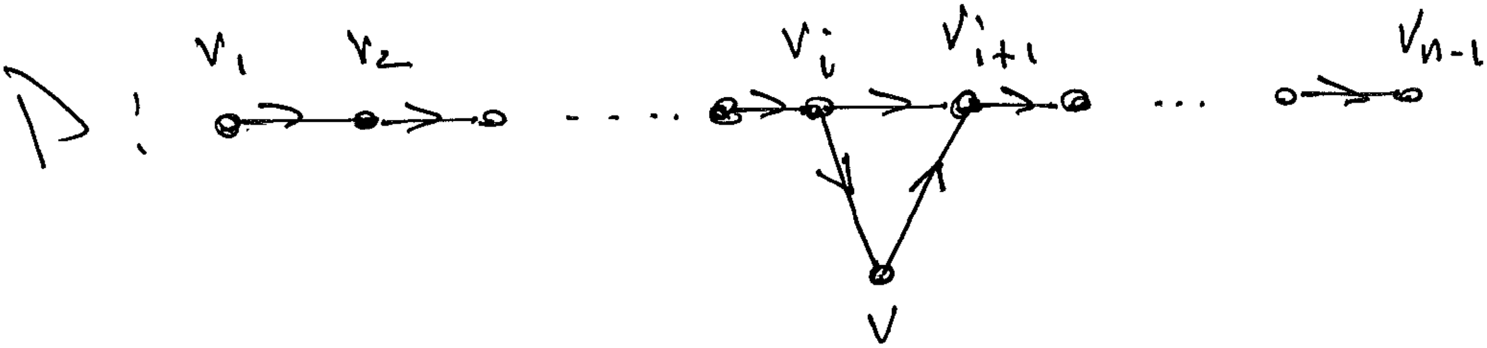
$$v \to v_1 \to \cdots \to v_{n-1}$$

which is also a Hamiltonian path in T.

Now assume $(v_{n-1}, v) \notin A(T)$ and $(v, v_1) \notin A(T)$. Then since T is a tournament, both $(v, v_{n-1}) \in A(T)$ and $(v_1, v) \in A(T)$.

LET v_i BE THE <u>LAST</u> VERTEX ALONG P WHICH IS THE ORIGIN OF A DIR ARC INTO v. WE HAVE $1 \le i < n-1$.

NECESSARILY v_{i+1} DOES NOT HAVE THIS PROPERTY, SO $(v, v_{i+1}) \in A(T)$



THEN

$$Q: v_1 \to v_2 \to \cdots \to v_i \to v \to v_{i+1} \to \cdots \to v_{n-1}$$

IS THE REQUIRED HAMILTONIAN PATH IN T. ///

THM

A strongly connected tournament T on n vertices ($n \geq 3$) contains dir cycles of length $3, 4, \ldots, n$.

COROLLARY (CAMION)

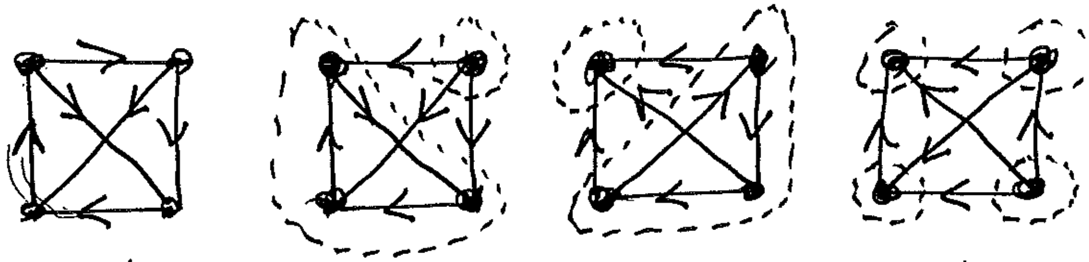
T is Hamiltonian iff it is strongly connected.

PROOF:

($\Leftarrow$) Follows directly from THM.

($\Rightarrow$) If $v_1 \rightarrow \cdots \rightarrow v_n \rightarrow v_1$ is a Ham. cycle in T, then obviously each v_i is reachable from each v_j. ///.

Ex. $n=4$



strongly conn. | 2 strong comp. | 2 strng. comp. | 4 strong comp.

Hamiltonian | Semi-Hamiltonian

Proof of Thm:

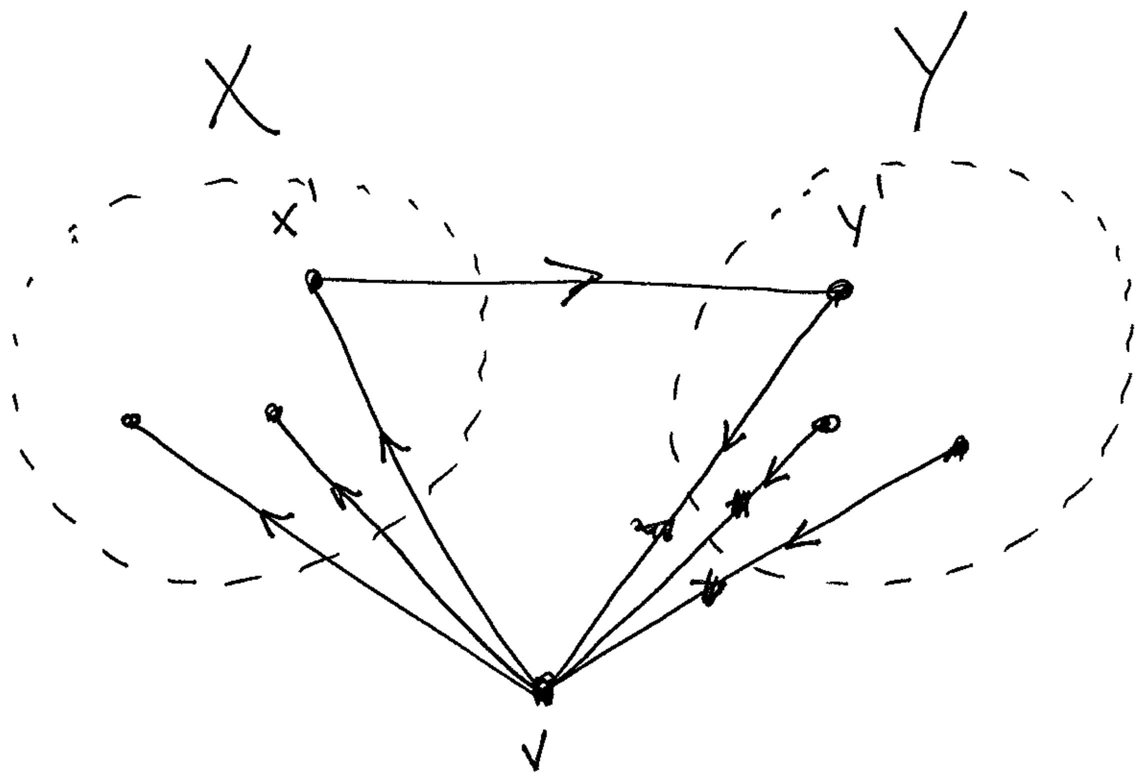
First show T contains a dir cycle of length 3, then proceed by induction to show T has dir cycles of length k for $3 \le k \le n$.

LET $v \in V(T)$. Define

$$X = \{ x \in V(T) \mid (v,x) \in A(T) \}$$

$$Y = \{ y \in V(T) \mid (y,v) \in A(T) \}$$

i.e. X, Y ARE 'losers' & 'winners' (resp.) AGAINST v.



NOTE: $X \cap Y = \phi$. Also $X \neq \phi$ AND $Y \neq \phi$ SINCE T is STRONGLY CONN.

ALSO, THERE MUST $x' \in X$ AND $y' \in Y$ WITH $(x', y') \in A(T)$ SINCE T IS STRONGLY CONN. (OTHERWISE NOTHING IN Y IS REACHABLE FROM ANYTHING IN X.)

Then $v \to x' \to y' \to v$ is A DIR. CYCLE of length 3.

LET $3 \leq k < n$, AND ASSUME T CONTAINS A DIR CYCLE OF length k. must show T CONTAINS A DIR CYCLE OF length $k+1$.

LET C BE THE ASSUMED CYCLE:

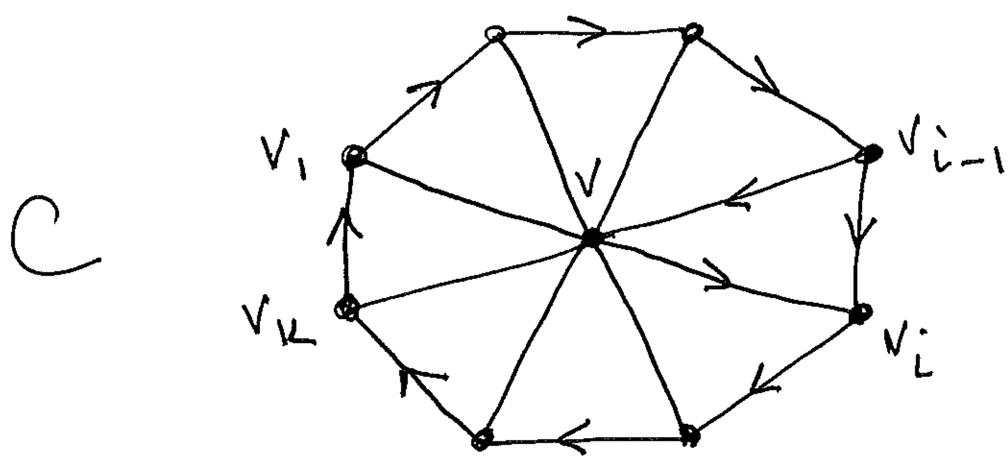
$C: v_1 \to v_2 \to \cdots \to v_k \to v_1$

HAVE 2 CASES:

<u>CASE 1</u>: THERE EXISTS $v \in V(T) - V(C)$ with ARCS Both to & from C

i.e. THERE ARE $v_i, v_j \in V(C)$ WITH $(v, v_i) \in A(T)$ and $(v_j, v) \in A(T)$.

In this CASE THERE EXISTS AN INDEX i $(1 < i \leq k)$ SUCH THAT $(v, v_i) \in A(T)$ AND $(v_{i-1}, v) \in A(T)$



SO WE HAVE THE DIR CYCLE

$$v_1 \to \cdots \to v_{i-1} \to v \to v_i \to \cdots \to v_k \to v_1$$

HAVING length $k+1$.

<u>CASE 2:</u>

NO SUCH VERTEX v EXISTS, i.e. ALL VERTICES NOT ON C HAVE ONLY OUTGOING ARCS TO, OR INCOMING ARCS FROM C.

THUS $V(T)-V(C)$ PARTITIONS INTO

$$X = \{x \in V(T) \mid (v_i, x) \in A(T) \text{ for } 1 \le i \le k\}$$

and

$$Y = \{y \in V(T) \mid (y, v_i) \in A(T) \text{ for } 1 \le i \le k\}$$

SO $V(T) = V(C) \cup X \cup Y$

IS A DISJOINT UNION.

OBSERVE BOTH $X \neq \phi$ & $Y \neq \phi$

SINCE T IS STRONGLY CONN.

~~&~~ AGAIN Since T is S.C.

MUST EXIST $x' \in X$, $y' \in Y$ WITH

$(x', y') \in A(T)$. THUS

$$v_1 \to x' \to y' \to v_3 \to \cdots \to v_k \to v_1$$

is REQ. cycle of length $k+1$.

///.