CMPE 177 8-3-09

LET G BE CONNECTED

Defn: A <u>HAMILTONIAN PATH</u> IN G IS A PATH WHICH INCLUDES EVERY VERTEX. A <u>HAMILTONIAN CYCLE</u> IS A CLOSED HAMILTONIAN PATH.

Recall: HAMILTONIAN GRAPH

SEMI-HAMILTONIAN GRAPH.

THEOREM (ORE)

LET G BE A SIMPLE GRAPH ON $n \geq 3$ VERTICES. SUPPOSE THAT FOR ALL $x, y \in V(G)$ WHICH ARE NON-ADJACENT, WE HAVE $\deg(x) + \deg(y) \geq n$. THEN G IS HAMILTONIAN.

COROLLARY (DIRAC)

LET G BE A SIMPLE GRAPH ON $n \geq 3$ VERTICES, AND SUPPOSE FOR ALL $x \in V(G)$ THAT $\deg(x) \geq \frac{n}{2}$. THEN G IS HAMILTONIAN.

PROOF:

FOR ANY PAIR OF VERTICES x, y (ADJACENT OR NOT) HAVE $\deg(x) + \deg(y) \geq \frac{n}{2} + \frac{n}{2} = n$, SO BY ORE'S THM, G IS HAMILTONIAN

PROOF OF ORE'S THM

SUPPOSE G SATISFIES THE CONDITION: FOR ANY NON-ADJACENT $x, y \in V(G)$:

$$\deg(x) + \deg(y) \geq n.$$

ASSUME, TO GET A CONTRADICTION, THAT G IS NOT HAMILTONIAN.

IF NECESSARY ADD EDGES TO G SO THAT IT REMAINS SIMPLE & NON-HAMILTONIAN. CONTINUE UNTIL THE ADDITION OF ONE MORE EDGE WOULD CREATE A HAM. CYC. CALL THIS GRAPH G', AND OBSERVE THAT THE DEGREE CONDITION IS TRUE IN G'.

ALSO NOTE, G' MUST CONTAIN A (NON-CLOSED) HAMILTONIAN PATH:

WHERE $V(G) = \{x_1, \ldots, x_n\}$ AND x_1 IS NOT ADJ. TO x_n.

THE DEGREE CONDITION GIVES

$$deg(x_1) + deg(x_n) \geq n .$$

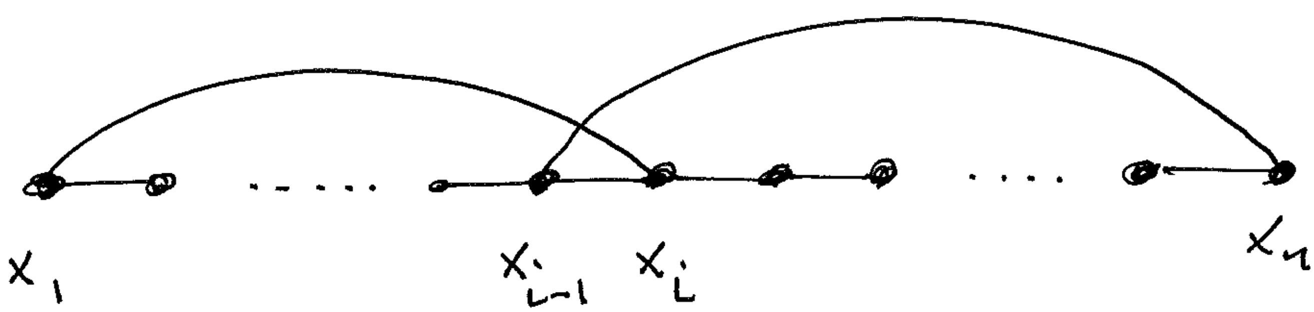
G' IS SIMPLE SO $deg(x)$ IS # OF NEIGHBORS OF x. THUS THERE ARE $deg(x_1)$ VERTICES x_i ADJ TO x_1, AND EACH OF THESE HAS A PREDECESSOR x_{i-1}. SUPPOSE EACH OF THESE PREDECESSORS IS NOT-ADJ TO x_n. THEN x_n HAS AT MOST $n-1-deg(x_1)$ MANY NEIGHBORS. THEN

$$deg(x_n) \leq n-1-deg(x_1)$$

$$\therefore \quad deg(x_1) + deg(x_n) \leq n-1$$

CONTRARY TO HYPOTHESIS. THUS

THERE MUST EXIST A CONSECUTIVE PAIR x_{i-1}, x_i SUCH THAT x_i IS ADJ TO x_1, AND x_{i-1} IS ADJACENT TO x_n.



BUT NOW G' CONTAINS THE HAM. CYC:

$$x_1, x_2, \dots x_{i-1}, x_n, x_{n-1}, \dots, x_i, x_1$$

CONTRADICTING THAT G' WAS NON-HAMILTONIAN. THUS OUR ASSUMPTION THAT G WAS NOT HAMILTONIAN WAS FALSE.

///.

THM

LET G BE SIMPLE. let $u, v \in V(G)$ BE NON-ADJACENT AND SUPPOSE

$$\deg(u) + \deg(v) \geq n$$

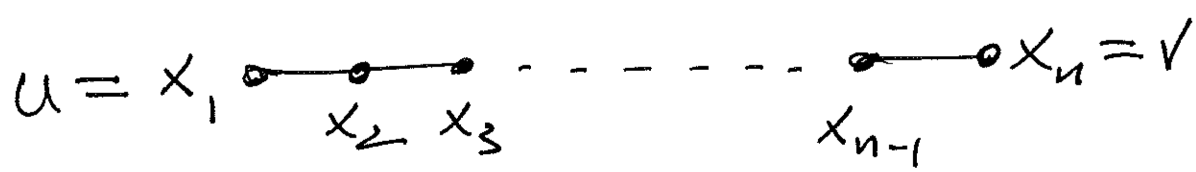
LET $G + uv$ DENOT ~~THE~~ (SUPER) GRAPH OBTAINED BY JOINING u TO v WITH A NEW EDGE. THEN G is HAMILTONIAN IFF $G + uv$ is HAMILTONIAN.

PROOF:

($\Rightarrow$) If G CONTAINS A HAM. CYC. THEN SO DOES $G + uv$.

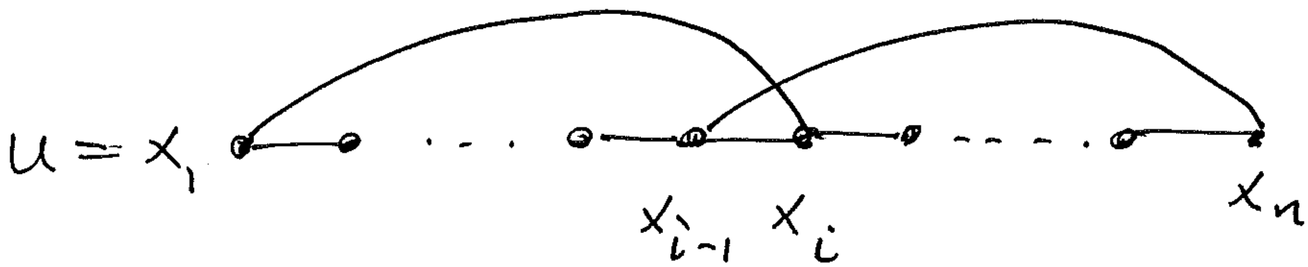
($\Leftarrow$) (LIKE PROOF OF ORE'S THM.)

SUPPOSE $G+uv$ IS HAMILTONIAN. ASSUME, TO GET A $\divideontimes$, THAT G IS NOT HAMILTONIAN. THEN THE HAM. CYC. IN $G+uv$ INCLUDES EDGE uv, SO G CONTAINS A (NON-CLOSED) HAMILTONIAN PATH:



JUST AS BEFORE x_1 MUST HAVE A NEIGHBOR x_i WHOSE PREDECESSOR x_{i-1} IS ADJ TO x_n. (OTHERWISE, AS BEFORE, $\deg(x_1) + d(x_n) \leq n-1$, CONTRARY TO HYPOTHESIS.)

THUS



$x_1, x_2, \dots, x_{i-1}, x_n, x_{n-1}, \dots, x_i, x_1$

IS A HAM. CYC. IN G, CONTRADICTING THAT G WAS NOT HAMILTONIAN.

$\therefore$ G <u>IS</u> HAMILTONIAN. ///

<u>Defn</u>

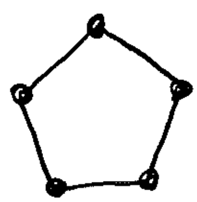
LET G BE SIMPLE ON n VERTICES. THE <u>CLOSURE</u> OF G IS THE SUPER GRAPH $c(G)$ OBTAINED AS FOLLOWS:

1.) while there exist non-adj $u, v \in V(G)$ such that $\deg(u) + \deg(v) \geq n$

2.) join u to v by a new edge.

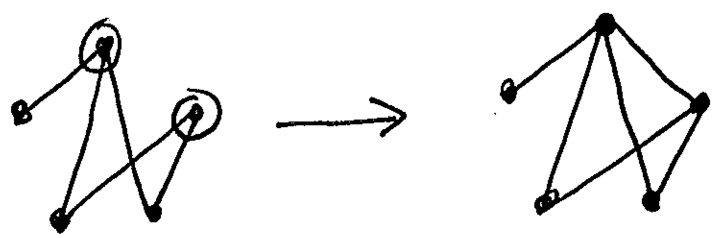
Examples ($n = 5$)

(a)



$G = c(G)$

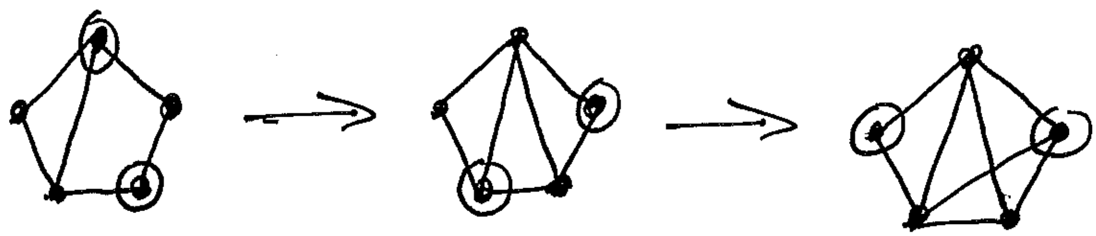
(b)

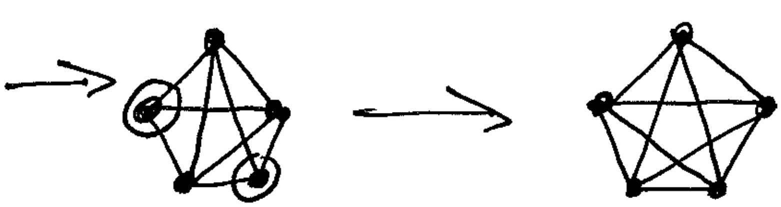


G $\qquad$ $c(G)$

(c)

G





$c(G) = K_5$

Corollary

A simple graph G is Hamiltonian iff $c(G)$ is Hamiltonian.

Proof: Exercise.

Corollary

Let G be a simple graph on $n \geq 3$ vertices. If $c(G) = K_n$, then G is Hamiltonian.

Proof: K_n is Hamiltonian for $n \geq 3$.

Note: Converse is false by Ex. (a) above.

(3.4) TRAVELLING SALESMAN PROBLEM

GIVEN A WEIGHTED GRAPH G,

DETERMINE

(1) IS G HAMILTONIAN ?

and

(2) IF SO, DETERMINE AN OPTIMAL HAMILTONIAN CYCLE IN G.

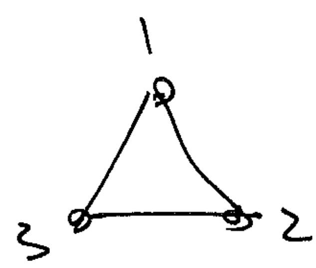
IF G IS COMPLETE, THEN THE ANSWER TO (1) IS YES.

WE CONSIDER WEIGHTED COMPLETE GRAPHS.

OF PERMUTATIONS OF n OBJECTS IS n!

EX $\{1, 2, 3\}$

$$\left.\begin{array}{c} 1\,2\,3 \\ 1\,3\,2 \\ 2\,1\,3 \\ 2\,3\,1 \\ 3\,1\,2 \\ 3\,2\,1 \end{array}\right\} 6 = 3!$$

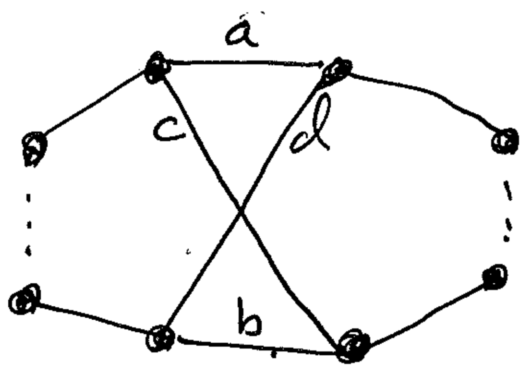


K_3

$$\left(\#\text{ of HAM. CYCS IN } K_n\right) = \frac{n!}{n \cdot 2} = \boxed{\frac{(n-1)!}{2}}$$

TWO OPTIMAL METHOD

GIVEN A HAMILTONIAN CYCLE C IN K_n AND 4 EDGES a, b, c, d



C

AS ABOVE SATISFYING :

$$w(c) + w(d) < w(a) + w(b)$$

WHEN SUCH EDGES CAN BE FOUND REPLACE

$$C' \leftarrow C - \{a, b\} + \{c, d\}$$

CONTINUE UNTIL NO SUCH EDGES EXIST.

EXERCISE:

CREATE AN EXAMPLE WHERE THE 2-OPTIMAL METHOD DOES NOT FIND A HAMILTONIAN CYCLE.

(7.1) DIRECTED GRAPHS (DIGRAPH)

Defn A DIGRAPH IS A PAIR $D = (V, A)$ OF SETS CALLED VERTICES ($V \neq \emptyset$) AND ARCS (A).

$$A = A(D) \subseteq V \times V.$$

NOTE: $A(D)$ CAN BE THOUGHT OF AS A RELATION ON V:

$$x A y \quad \text{iff} \quad (x, y) \in A(D).$$

If $a = (u, v) \in A(D)$

$$u \xrightarrow{a} v$$

WE SAY a IS AN ARC from u to v, AND CALL u the origin & v the terminus of a.

GIVEN A DIGRAPH D, THE <u>UNDERLYING GRAPH</u> G, IS OBTAINED BY REPLACING EACH $(u,v) \in A(D)$ BY THE EDGE $\{u,v\} \in E(G)$.

GIVEN A GRAPH G, AN <u>ORIENTATION</u> OF G IS A DIGRAPH D WHOSE UNDERLYING GRAPH IS G.

HAVE DIRECTED ANALOGUES OF: <u>WALK</u>, <u>TRAIL</u>, <u>PATH</u>, <u>CYCLE</u>, ETC...

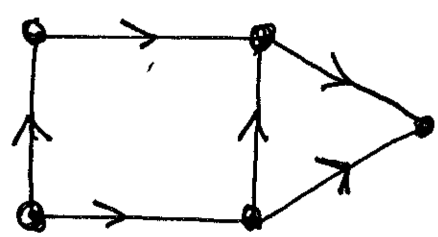
WE SAY $v \in V(D)$ IS <u>REACHABLE</u> from $u \in V(D)$ IFF D CONTAINS A (DIRECTED) u-v PATH.

ORIGIN TERMINUS

<u>Defn</u> D is called <u>weakly connected</u> iff its underlying graph is connected.

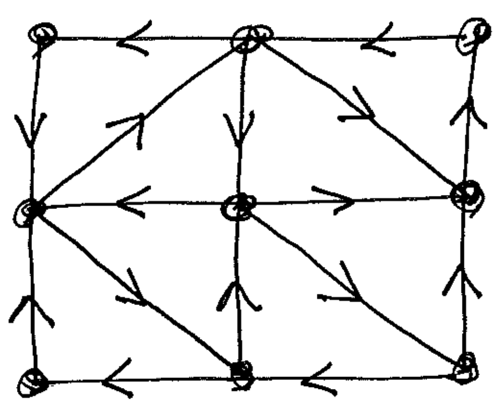
<u>Defn</u> D is called <u>strongly connected</u> iff each $v \in V(D)$ is reachable from each $u \in V(D)$.

<u>Ex.</u>



weakly conn. but not strongly conn.

<u>Ex.</u>



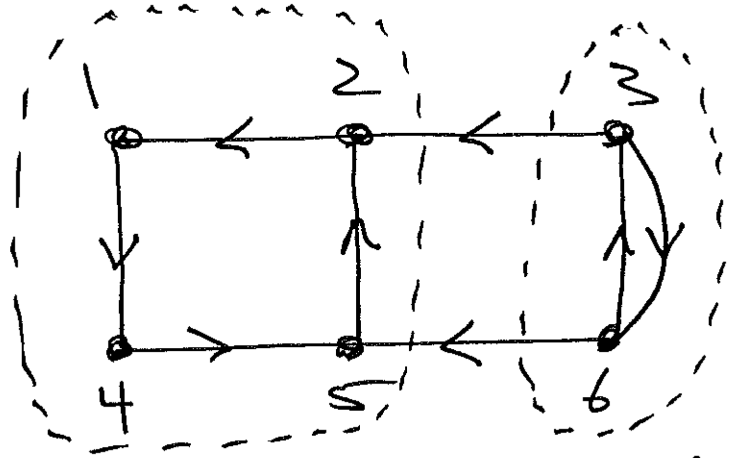
strongly conn.

Defn
A SUB-DIGRAPH OF $D = (N, A)$ is
A DIGRAPH $D' = (V', A')$ SUCH
THAT $V' \subseteq V$ and $A' \subseteq A$.

Defn A STRONGLY CONNECTED COMPONENT
(SCC) OF D is A STRONGLY CONN.
SUB-DIGRAPH S SUCH THAT
NO SUBDIGRAPH OF D WHICH PROPERLY
CONTAINS S is ALSO STRONGLY CONN.

i.e. S is MAXIMAL W.R.T. THE
PROPERTY OF BEING STRONGLY CONN.

EX.



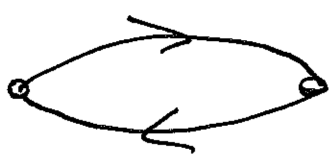
STRONG COMPONENTS ARE INDUCED BY : $\{3, 6\}$ and $\{1, 2, 4, 5\}$.

Defn

D IS CALLED SIMPLE IFF IT HAS

NO PARALLEL ARCS 

AND NO DIR LOOPS 

NOTE:  IS NOT CONSIDERED PARALLEL ARCS

<u>Defn</u> Two Digraphs D_1, D_2 are said to be <u>isomorphic</u> iff there exist bijections

$$f : V(D_1) \to V(D_2)$$
$$g : A(D_1) \to A(D_2)$$

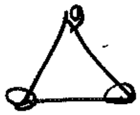
such that for all $(u,v) \in A(D_1)$

$$g(u,v) = (f(u), f(v)) \in A(D_2)$$

It follows that f^{-1}, g^{-1} satisfy same property: for $(x,y) \in A(D_2)$

$$g^{-1}(x,y) = (f^{-1}(x), f^{-1}(y)).$$

Write $D_1 \cong D_2$ to mean D_1 & D_2 are isomorphic.

Ex. K_3  HAS 8 ORIENTATIONS

D_1  D_5 

D_2  D_6 

D_3  D_7 

D_4  D_8 

NOTICE: $D_1 \cong D_8$

$D_2 \cong D_3 \cong \cdots\cdots \cong D_7$

<u>EXERCISE</u>: HOW MANY ORIENTATIONS HAS K_n

(HINT: RECALL $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$)

<u>ANS</u>: $2^{\binom{n}{2}}$

<u>EXERCISE</u>: DRAW ALL ORIENTATIONS ON K_4.

<u>EXERCISE</u>: (HARD!)

UP TO ISOMORPHISM, HOW MANY DIGRAPHS HAVE K_n AS UNDERLYING GRAPH?