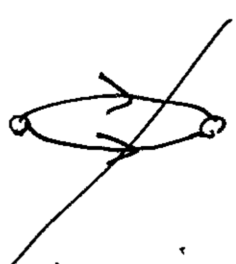
CMPE 177
8-10-09

.../Summer09

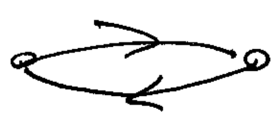
/graphs /randplay

random_graph

Ext. Project sourcecode submission to Fri. 8/14/09 10:00 pm.



not simple



simple

8.1 Flows & Cuts

A NETWORK N is a (weakly) conn. simple digraph together with an integer valued CAPACITY fcn.

$$c : A(N) \longrightarrow \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$$

$c(a)$ is the capacity of $a \in A(N)$.

We call $s \in V(N)$ a SOURCE if $id(s) = 0$, and call $t \in V(N)$ a sink if $od(t) = 0$.

Other vertices are called INTERMEDIATE.

ASSUME: N contains one source s, & one sink t.

For $u \in V(N)$ write

$I(u) = \{\text{arcs having } u \text{ as terminus}\} \subseteq A(N)$

$O(u) = \{\text{arcs having } u \text{ as origin.}\} \subseteq A(N)$.

<u>Defn</u> A <u>flow</u> (or <u>feasible flow</u>) in N is a fcn.

$$f : A(N) \to \mathbb{Z}_+$$

s.t.

(i) <u>Capacity Constraint</u> : $f(a) \leq c(a)$ for all $a \in A(N)$,

(ii) outflow from s = inflow to t

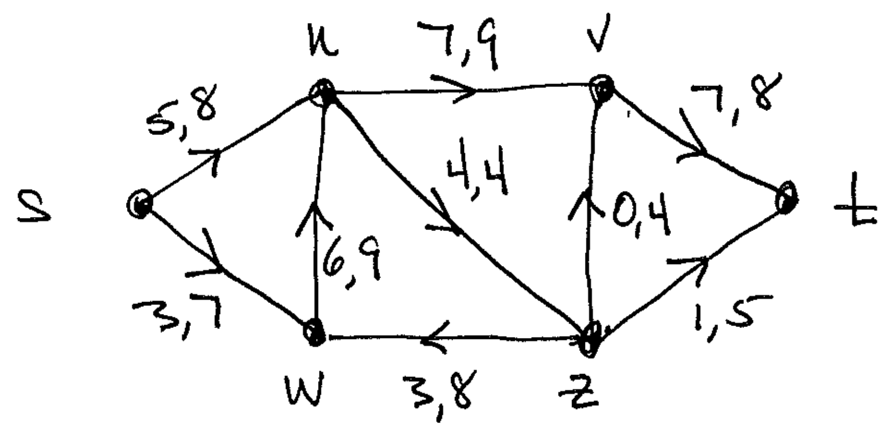
$$\sum_{a \in O(s)} f(a) = \sum_{a \in I(t)} f(a)$$

(iii) Flow Conservation

for all INTERMEDIATE $x \in V(N)$
outflow from x = inflow to x

$$\sum_{a \in O(x)} f(a) = \sum_{a \in I(x)} f(a)$$

Ex. 1



$d = 8$

labels (f, c). Define

$$d = \sum_{a \in O(s)} f(a) = \sum_{a \in I(t)} f(a)$$

we call d the value of the flow f.

Defn A <u>CUT</u> in N is a set of arcs of the form $A(X, \bar{X})$ where $s \in X$ and $t \in \bar{X}$.

Ex 1: $X = \{s, u, w\}$, $\bar{X} = \{v, z, t\}$

$A(X, \bar{X}) = \{uv, uz\}$ <u>is</u> a CUT.

$c(X, \bar{X}) = 9 + 4 = 13$

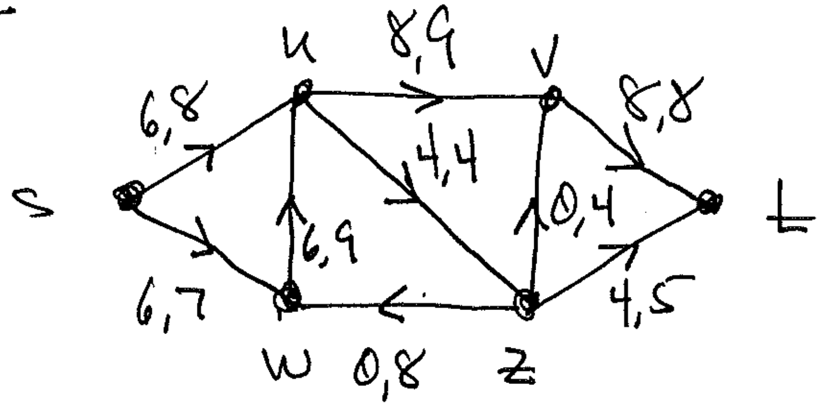
$f(X, \bar{X}) = 7 + 4 = 11$

$A(\bar{X}, X) = \{zw\}$ <u>not</u> a cut.

$c(\bar{X}, X) = 8$

$f(\bar{X}, X) = 3$

Ex 2.



$d = 12$

NOTATION: Given $X, Y \subseteq V(N)$ write

$$A(X,Y) = \{\text{arcs having origin in } X, \text{ terminus in } Y\}$$

Given any $g : A(N) \to \mathbb{Z}_+$, write

$$g(X,Y) = \sum_{a \in A(X,Y)} g(a).$$

If $X \subseteq V(N)$ write $\overline{X} = V(N) - X$.

Ex. 2 $X = \{s, u, v, w\}$, $\bar{X} = \{z, t\}$

$A(X, \bar{X}) = \{uz, vt\}$ is a cut.

$c(X, \bar{X}) = 4 + 8 = 12$

$f(X, \bar{X}) = 4 + 8 = 12$

Also

$A(\bar{X}, X) = \{zv, zw\}$

$c(\bar{X}, X) = 4 + 8 = 12$

$f(\bar{X}, X) = 0$

THM

Suppose f is a flow on N with value d, AND $A(X, \bar{X})$ is any cut.
THEN

$$d = f(X, \bar{X}) - f(\bar{X}, X) \le c(X, \bar{X}).$$

i.e. d is the 'net' flow across any cut, d is BOUNDED ABOVE BY $c(X, \bar{X})$.

PROOF:

Since f is a flow have

$$f(\{s\}, V) = d \quad \text{and} \quad f(V, \{s\}) = 0$$

for any intermediate $u \in V(N)$

$$f(\{u\}, V) = \sum_{a \in O(u)} f(a) = \sum_{a \in I(u)} f(a) = f(V, \{u\})$$

Thus

$$f(\{u\}, V) - f(V, \{u\}) = 0$$

If $A(X, \bar{X})$ is a cut, then

$$f(X, V) - f(V, X)$$

$$= \sum_{x \in V(N)} \left(f(\{x\}, V) - f(V, \{x\}) \right)$$

$$= f(\{s\}, V) - f(V, \{s\})$$

$$= d \quad .$$

But

$$f(X,V) = f(X, X \cup \bar{X})$$
$$= f(X,X) + f(X,\bar{X})$$

and

$$f(V,X) = f(X \cup \bar{X}, X)$$
$$= f(X,X) + f(\bar{X},X)$$

Thus

$$f(X,\bar{X}) - f(\bar{X},X) = f(X,V) - f(V,X)$$
$$= d.$$

Since $f(a) \le c(a)$ for all $a \in A(N)$, we have

$$f(X,\bar{X}) \le c(X,\bar{X})$$

so

$$f(X,\bar{X}) - f(\bar{X},X) \le f(X,\bar{X}) \le c(X,\bar{X}).$$

So $d = f(X,\bar{X}) - f(\bar{X},X) \le c(X,\bar{X})$.

Note: Value d of any flow satisfies

$$d \le \min\{ c(X,\bar{X}) \mid A(X,\bar{X}) \text{ is a cut} \}$$

Defn

A flow f for which equality is called a maximum flow.

Ex 2: Had $d = 12 = c(X,\bar{X})$, so, f is a maximum flow. Also $A(X,\bar{X})$ is a minimum capacity cut.

NOTE: if $|V(N)| = n$ (so $n-2$ intermediate vertices) then N contains 2^{n-2} cuts, one for each subset of {interm. vert.}.

Note: $d_f \le c(X,\bar{X})$
Holds for <u>any</u> flow f, and <u>any</u> cut $A(X,\bar{X})$.

<u>Lemma</u>
If f is a flow with valued and $A(X,\bar{X})$ is a cut, and if

$$d = c(X,\bar{X})$$

then f is maximum and $A(X,\bar{X})$ is a minimum capacity cut.

<u>Theorem</u> (max flow - min cut)
Let N be a capacitated network with capacity c, then there exists a max flow with value

$$d = \min\{c(X,\bar{X}) \mid A(X,\bar{X}) \text{ is a cut}\}.$$

Let $T: v_0 v_1 \dots v_k$ BE A (NOT NECESSARILY DIRECTED) TRAIL IN N.

- We call $v_{i-1}v_i$ A <u>FORWARD ARC</u> of T iff $(v_{i-1}, v_i) \in A(N)$.
- We call $v_{i-1}v_i$ A <u>Reverse Arc</u> of T iff $(v_i, v_{i-1}) \in A(N)$

Let f BE A flow in N.
We ASSOCIATE To each $a \in A(N)$ $i(a)$ called <u>increment</u> of a:

$$i(a) = \begin{cases} c(a) - f(a) & \text{if } a \text{ forward in } T \\ f(a) & \text{if } a \text{ reverse in } T \end{cases}$$

WRITE $i(a) = i_T(a)$ TO emphasize dependence on T.

NOTE $i(a)$ is amount by which f could be increased on a forward arc and still satisfy $f(a) \le c(a)$, and amount by which f could be decreased on a reverse arc and still satisfy $f(a) \ge 0$.

<u>Defn</u> The INCREMENT of f ALONG $\vec{T}$ is

$$i(\vec{T}) = \min\{ i(a) \mid a \text{ is an arc in } \vec{T} \}$$

Sometimes write $i(\vec{T}) = i_f(\vec{T})$

<u>Defn</u>
We say $\vec{T}$ is <u>f-saturated</u> iff $i(\vec{T}) = 0$, and <u>f-UNSATURATED</u> iff $i(\vec{T}) > 0$.

<u>Defn</u>
T is an <u>f-incrementing</u> (also <u>f-AUGMENTING</u>) TRAIL iff T goes from s to t and is f-unsaturated

$$T: s = v_0, v_1, \ldots, v_k = t$$

<u>Lemma</u>
LET T BE AN f-AUGMENTING TRAIL, THEN the fcn $f_1 : A(N) \to \mathbb{Z}_+$ Given by

$$f_1(a) = \begin{cases} f(a) + i(T) & \text{if } a \text{ forward in } T \\ f(a) - i(T) & \text{if } a \text{ reverse in } T \\ f(a) & \text{if } a \text{ not in } T \end{cases}$$

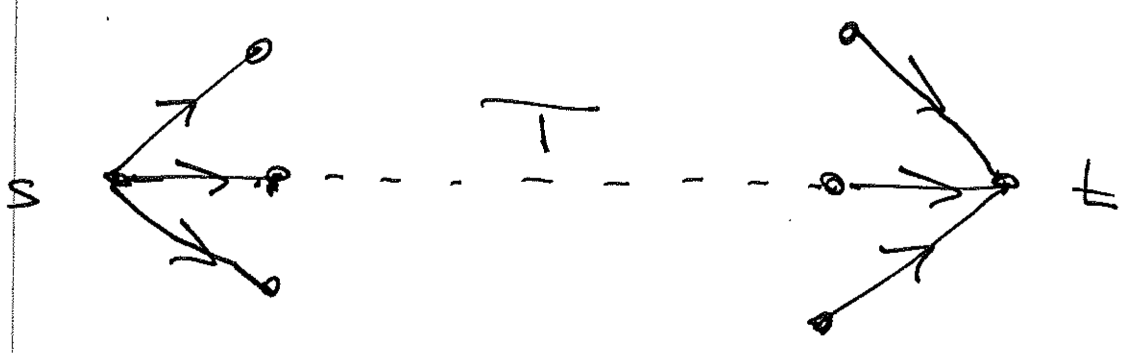
is a flow in N. If f has value d, then f_1 has value $d + i_f(T)$, AND T is f_1-SATURATED.

<u>PROOF:</u>

OBSERVE 1ST THAT $0 \le f_1(a) \le c(a)$ for all $a \in A(N)$. So (i) of Defn of flow is satisfied.

If T is incident with s just once, THEN

$$\sum_{a \in O(s)} f_1(a) = \left(\sum_{a \in O(s)} f(a) \right) + i_f(T)$$
$$= d + i_f(T)$$



If T is incident with s more than once, then since $id(s) = 0$, THE contributions of $i(T)$ to $\sum_{a \in O(a)} f_1(a)$ along for. & rev. arcs cancel.

So again

$$\sum_{a\in O(s)} f_1(a) = \left(\sum_{a\in O(s)} f(s)\right) + i_f(T)$$

$$= d + i_f(T).$$

Similarly we have

$$\sum_{a\in I(t)} f_1(a) = \left(\sum_{a\in I(t)} f(a)\right) + i_f(T)$$

$$= d + i_f(T)$$

Thus (ii) from defn of flow holds;

$$\sum_{a\in O(s)} f_1(a) = \sum_{a\in I(t)} f_1(a).$$

Now let $x \in V(N)$ be intermed.
Since f is a flow;

$$(*) \quad \sum_{a\in O(x)} f(a) = \sum_{a\in I(x)} f(a).$$

If x is NOT INCIDENT WITH T,
then

$$(**) \quad \sum_{a \in O(x)} f_1(a) = \sum_{a \in I(x)} f_1(a) \; .$$

so (iii) HOLDS IN THIS CASE.
If x is INCIDENT with T
THEN have 4 Cases:

(1) $\cdots\; Y \to X \to Z \;\cdots \quad T$

we add $i_f(T)$ to Both LHS &
RHS of $(*)$ to get $(**)$

(2) $\cdots\; Y \leftarrow X \leftarrow Z \;\cdots \quad T$

we sustract $i_f(T)$ from Both LHS
RHS OF $(*)$ to get $(**)$

(3) $\cdots\; Y \to X \leftarrow Z \;\cdots \quad T$

we add $i_f(T)$ to one term &
sustract $i_f(T)$ from another in RHS
of $(*)$ so again $**$ holds.

(4) $\cdots\; Y \longleftarrow X \longrightarrow Z \;\cdots\; T$

AGAIN $i_f(T)$ is both added & subtracted to LHS of (*) to get **.

In all cases we have (iii)

$$\sum_{a \in O(x)} f_1(a) = \sum_{a \in I(x)} f_1(a)$$

Also showed in (ii) that value of f_1 is

$$d + i_f(T)$$

CHECK: T is f_1-SATURATED

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