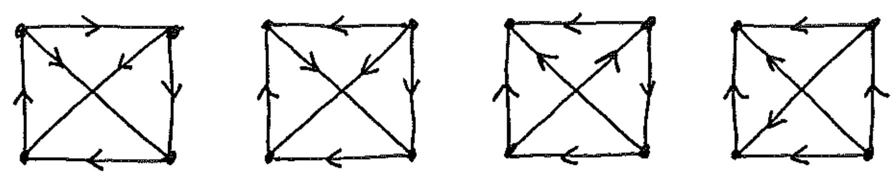
7.3 TOURNAMENTS.

A TOURNAMENT is an orientation of K_n.

There are $2^{\binom{n}{2}}$ distinct orientations of K_n, but up to isomorphism, there are far fewer tournaments on n vertices.

EX.
K_4 has $2^6 = 64$ orientations, but there are only 4 non-isomorphic tournaments with underlying graph K_4:



EXERCISE
There are 12 non-isomorphic tournaments on 5 vertices. Draw them.

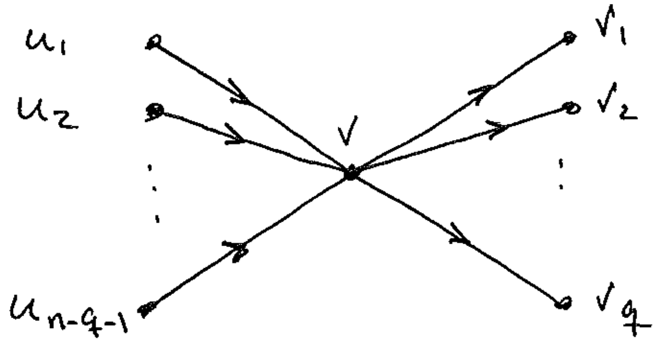
These digraphs are called tournaments since they can be used to record the results of a round-robin competition (every player plays every other player once.) A directed edge $u \bullet\!\!\rightarrow\!\!\bullet v$ means that u beat v;

THEOREM

LET v BE A VERTEX OF LARGEST OUTDEGREE IN A TOURNAMENT T (i.e. v IS A "WINNER"), THEN FOR EVERY OTHER VERTEX u IN T, THERE IS A DIRECTED v-u PATH OF LENGTH AT MOST 2.

PROOF:

LET $q = od(v)$ AND LET $\{v_1, \ldots, v_q\}$ BE THE VERTICES JOINED BY AN ARC FROM v. SINCE T IS A TOURNAMENT (SAY ON K_n) THE REMAINING $n-q-1$ VERTICES $\{u_1, \ldots, u_{n-q-1}\}$ ARE JOINED BY ARCS TO v.



THERE ARE DIRECTED PATHS OF LENGTH 1 FROM v TO v_i FOR $1 \leq i \leq q$.

IT REMAINS TO SHOW THERE ARE DIRECTED PATHS OF LENGTH 2 FROM v TO u_j FOR $1 \leq j \leq n-q-1$.

GIVEN SUCH A u_j, IF THERE IS AN ARC FROM SOME v_i TO u_j, THEN $v \to v_i \to u_j$ IS THE REQUIRED PATH OF LENGTH 2. WE SHOW NOW THAT FOR EVERY u_j THERE IS SUCH A v_i.

SUPPOSE, TO GET A CONTRADICTION, THAT THERE IS A u_k $(1 \le k \le n-q-1)$ WITH NO INCOMING ARC $v_i \to u_k$, FOR ANY i, $1 \le i \le q$.

SINCE T IS A TOURNAMENT, THERE MUST BE ARCS $u_k \to v_i$ FOR ALL i, $1 \le i \le q$. BUT THEN

$$od(u_k) = q+1$$

CONTRADICTING OUR CHOICE OF v AS HAVING MAXIMUM OUTDEGREE.

THIS CONTRADICTION SHOWS THAT NO SUCH u_k EXISTS, AND COMPLETES THE PROOF.

///

DEFN.
A DIGRAPH D IS CALLED HAMILTONIAN IF IT CONTAINS A CLOSED DIRECTED TRAIL WHICH VISITS EACH VERTEX EXACTLY ONCE.

SUCH A TRAIL IS NECESSARILY A CYCLE CALLED A (DIRECTED) HAMILTONIAN CYCLE.

A (NOT NECESSARILY CLOSED) DIRECTED PATH IN D WHICH VISITS EACH VERTEX EXACTLY ONCE IS CALLED A HAMILTONIAN PATH.

A NON-HAMILTONIAN DIGRAPH WHICH CONTAINS A (NECESSARILY OPEN) HAMILTONIAN PATH IS CALLED SEMI-HAMILTONIAN.

THEOREM (RÉDEI)
EVERY TOURNAMENT T CONTAINS A DIRECTED HAMILTONIAN PATH.

REMARK:
IN OTHER WORDS, EVERY TOURNAMENT IS EITHER HAMILTONIAN OR SEMI-HAMILTONIAN.

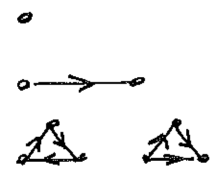
PROOF:
WE USE INDUCTION ON $n = |V(T)|$.

ANY OF THE EXAMPLES

$n=1$

$n=2$

$n=3$



CAN SERVE AS THE BASE CASE.

SUPPOSE $n=|V(T)|\geq 3$, AND ASSUME THAT ANY TOURNAMENT ON $n-1$ VERTICES CONTAINS A (DIRECTED) HAMILTONIAN PATH.

PICK $v\in V(T)$. OBSERVE THAT $T-v$ IS A TOURNAMENT ON $n-1$ VERTICES, SO BY OUR INDUCTION HYPOTHESIS, CONTAINS A HAMILTONIAN PATH

$$P:\ v_1\to v_2\to \cdots \to v_{n-1}$$

IF $(v_{n-1},v)\in A(T)$ THEN P CAN BE EXTENDED TO

$$v_1\to \cdots \to v_{n-1}\to v$$

which is A HAMILTONIAN PATH IN T. LIKEWISE, IF $(v,v_1)\in A(T)$ THEN

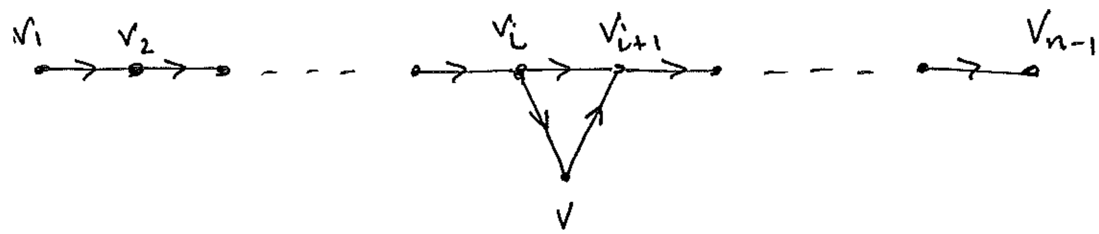
$$v\to v_1\to \cdots \to v_{n-1}$$

IS A HAMILTONIAN PATH.

ASSUME THAT NEITHER $(v_{n-1}, v) \in A(T)$ NOR $(v, v_1) \in A(T)$. SINCE T IS A TOURNAMENT (I.E. ITS UNDERLYING GRAPH IS COMPLETE) WE MUST HAVE $(v_1, v) \in A(T)$.

LET v_i BE THE LAST VERTEX ALONG P WHICH IS THE ORIGIN OF A DIRECTED ARC <u>TO</u> v. BY OUR ASSUMPTION $1 \le i < n-1$.

THEN NECESSARILY v_{i+1} DOES NOT HAVE THIS PROPERTY, WHENCE $(v, v_{i+1}) \in A(T)$.



THEN $Q: v_1 \to \cdots \to v_i \to v \to v_{i+1} \to \cdots \to v_{n-1}$ IS THE REQUIRED HAMILTONIAN PATH IN T.

THE RESULT FOLLOWS BY INDUCTION.

///

THEOREM

A STRONGLY CONNECTED TOURNAMENT T ON n VERTICES CONTAINS DIRECTED CYCLES OF LENGTH $3, 4, \ldots, n$.

COROLLARY (CAMION)

T IS HAMILTONIAN IF AND ONLY IF IT IS STRONGLY CONNECTED.

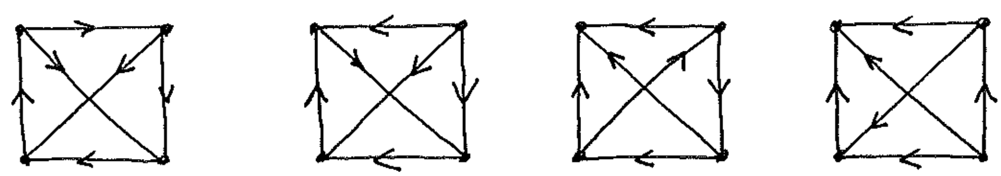
PROOF.

($\Rightarrow$) IF $v_1 \to v_2 \to \cdots \to v_n \to v_1$ IS A HAMILTONIAN CYCLE IN T THEN OBVIOUSLY EACH v_i IS REACHABLE FROM EACH v_j.

($\Leftarrow$) THIS IS JUST A SPECIAL CASE OF THE PRECEDING THEOREM.

///.

EX. $n = 4$



STRONGLY CONNECTED — 2 STRONG COMPONENTS — 2 STRONG COMPONENTS — 4 STRONG COMPONENTS

HAMILTONIAN — $\underbrace{\text{NOT HAMILTONIAN}}$ (BUT SEMI-HAMILTONIAN)

PROOF OF THEOREM:

WE FIRST SHOW T CONTAINS A DIRECTED CYCLE OF LENGTH 3, THEN PROCEDE BY (FINITE) INDUCTION TO SHOW T HAS DIRECTED CYCLES OF LENGTH k FOR $3 \le k \le n$.

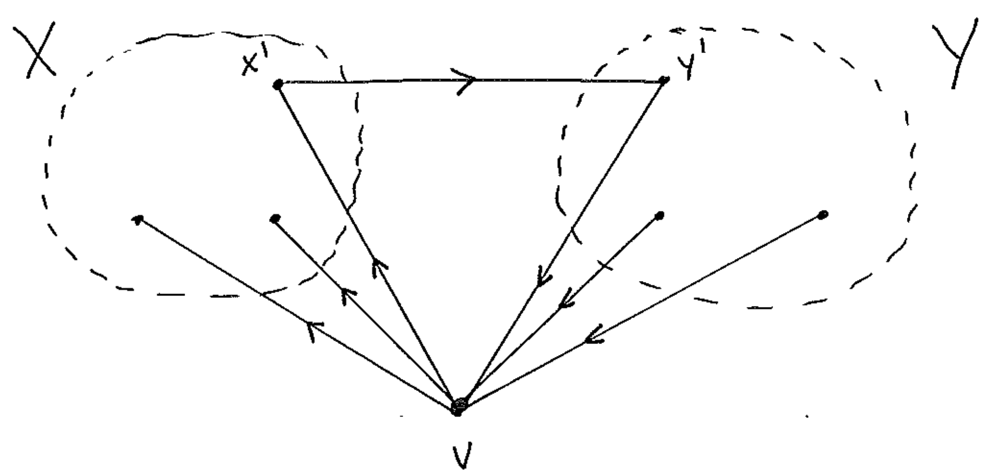
LET $v \in V(T)$ AND DEFINE VERTEX SETS

$$X = \{x \in V(T) : (v,x) \in A(T)\}$$

AND

$$Y = \{y \in V(T) : (y,v) \in A(T)\}.$$

i.e. X AND Y ARE THE SETS OF "LOSERS" AND "WINNERS" RESPECTIVELY, AGAINST v.



OBSERVE THAT $X \cap Y = \emptyset$ SINCE T IS A TOURNAMENT, AND $X \neq \emptyset$, $Y \neq \emptyset$ SINCE T IS STRONGLY CONNECTED.

ALSO THERE MUST EXIST $x' \in X$ AND $y' \in Y$ WITH $(x', y') \in A(T)$, AGAIN SINCE T IS STRONGLY. (IF NOT NO VERTEX IN X IS REACHABLE FROM ANY VERTEX IN Y.)

$v \to x' \to y' \to v$ IS THE REQUIRED CYCLE OF LENGTH 3.

NOW LET $3 \leq k < n$, AND ASSUME T CONTAINS A DIRECTED CYCLE OF LENGTH k. WE SHOW T ALSO CONTAINS A DIRECTED CYCLE OF LENGTH $k+1$.

LET C BE THE ASSUMED CYCLE.

$$C: v_1 \to v_2 \to \dots \to v_k \to v_1$$

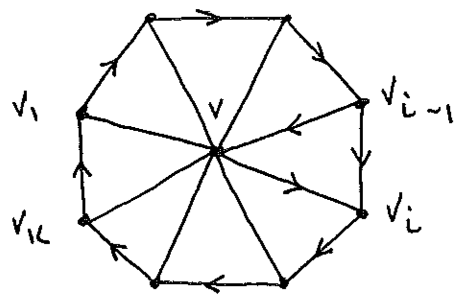
THERE ARE TWO CASES TO CONSIDER.

CASE 1:

THERE EXISTS A VERTEX v NOT ON C WITH ARCS BOTH TO AND FROM C. i.e. C CONTAINS VERTICES v_i, v_j WITH $(v, v_i) \in A(T)$ AND $(v_j, v) \in A(T)$.

IN THIS CASE THERE MUST EXIST AN INDEX i ($1 < i \leq k$) SUCH THAT $(v, v_i) \in A(T)$ AND $(v_{i-1}, v) \in A(T)$.

C



THEN $v_1 \rightarrow \cdots \rightarrow v_{i-1} \rightarrow v \rightarrow v_i \rightarrow \cdots \rightarrow v_k \rightarrow v_1$ IS THE REQUIRED CYCLE OF LENGTH $k+1$.

<u>CASE 2.</u>

NO SUCH VERTEX v EXISTS, i.e. ALL VERTICES NOT ON C HAVE EITHER ONLY OUTGOING ARCS TO C, OR ONLY INCOMING ARCS FROM C.

THUS $V(T) - V(C)$ PARTITIONS INTO TWO SETS

$$X = \{x \in V(T) : (v_i, x) \in A(T) \text{ FOR } i = 1, .., k\}$$

AND

$$Y = \{y \in V(T) : (y, v_i) \in A(T) \text{ FOR } i = 1, .., k\}.$$

i.e. $V(T) = V(C) \cup X \cup Y$ IS A DISJOINT UNION.

OBSERVE THAT NEITHER $X = \phi$ NOR $Y = \phi$ SINCE T IS STRONGLY CONNECTED.

(WE CANNOT HAVE BOTH $X = \phi$ AND $Y = \phi$ SINCE $k < n$. IF $X = \phi$ AND $Y \neq \phi$ THEN NO VERTEX IN Y IS REACHABLE FROM ANY VERTEX IN C. IF $X \neq \phi$ AND $Y = \phi$ THEN NO VERTEX IN C IS REACHABLE FROM ANY VERTEX IN X.)

AGAIN SINCE T IS STRONGLY CONNECTED, THERE MUST EXIST $x' \in X$, $y' \in Y$ WITH $(x', y') \in A(T)$. (OTHERWISE NO VERTEX IN Y IS REACHABLE FROM ANY VERTEX IN X.)

IN THIS CASE

$$v_1 \to x' \to y' \to v_3 \to \cdots \to v_k \to v_1$$

IS THE REQUIRED CYCLE OF LENGTH $k+1$, AND THE INDUCTION IS COMPLETE.

///