7.2 DEGREES

THE INDEGREE $id(v)$ OF A VERTEX $v \in V(D)$ IS THE NUMBER OF ARCS HAVING v AS TERMINUS.

THE OUTDEGREE $od(v)$ IS THE NUMBER OF ARCS HAVING v AS ORIGIN.

THE DEGREE $d(v)$ IS THE DEGREE OF v IN THE UNDERLYING GRAPH OF D. THUS

$$d(v) = id(v) + od(v).$$

EXERCISE

DETERMINE HOW TO READ OFF THE INDEGREES AND OUTDEGREES FROM THE ADJACENCY AND INCIDENCE MATRICES.

THEOREM (BACK-SLAP LEMMA)

LET $V(D) = \{v_1, \ldots, v_n\}$. THEN

$$\sum_{i=1}^{n} id(v_i) = \sum_{i=1}^{n} od(v_i) = |A(D)|.$$

PROOF.

EACH ARC CONTRIBUTES EXACTLY 1 TO THE INDEGREE OF ITS TERMINUS, AND EXACTLY 1 TO THE OUTDEGREE OF ITS ORIGIN. ///

WE CONSIDER EULERIAN TRAILS AND SEMI-EULERIAN TRAILS IN DIGRAPHS, WITH OBVIOUS DEFINITIONS.

THE BOOK CALLS THESE EULER TOURS AND EULER TRAILS RESPECTIVELY. WE WILL USE THE TERM EULERIAN CIRCUIT FOR A CLOSED DIRECTED TRAIL WHICH TRAVERSES EACH EDGE EXACTLY ONCE.

THUS D IS CALLED EULERIAN IF IT CONTAINS AN EULERIAN CIRCUIT, SEMI-EULERIAN IF IT CONTAINS AN (OPEN) EULERIAN TRAIL, BUT NO CIRCUIT.

THEOREM
LET D BE (WEAKLY) CONNECTED, WITH AT LEAST ONE ARC. THEN D IS EULERIAN IF AND ONLY IF

$$id(v) = od(v)$$

FOR ALL $v \in V(D)$.

PROOF.
($\Rightarrow$) WHEN AN EULERIAN CIRCUIT PASSES THROUGH $v \in V(D)$ IT CONTRIBUTES 1 TO THE INDEGREE

OF v, AND 1 TO THE OUTDEGREE. SINCE EACH ARC BELONGS TO THIS TRAIL, $id(v)$ AND $od(v)$ ARE THE SUMS OF THESE CONTRIBUTIONS. $\therefore\ id(v) = od(v)$.

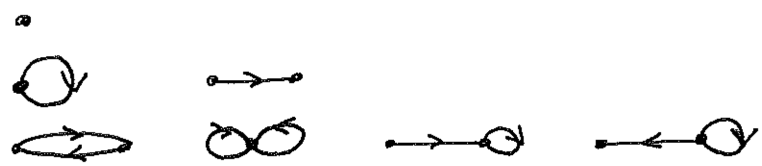
($\Leftarrow$) SUPPOSE THAT $id(v) = od(v)$ FOR ALL VERTICES v. WE USE INDUCTION ON THE NUMBER OF ARCS $m = |A(D)|$.

WE CAN TAKE ANY OF THE EXAMPLES BELOW AS BASE CASE

$m = 0$

$m = 1$

$m = 2$



LET $m > 1$ AND ASSUME THAT ANY DIGRAPH WITH FEWER THAN m ARCS, AND WITH EQUAL INDEGREES AND OUTDEGREES, CONTAINS AN EULERIAN CIRCUIT.

LET T BE ANY DIRECTED TRAIL IN D, SAY FROM u TO v. SINCE $id(v) = od(v) > 0$ (D BEING WEAKLY CONNECTED), THERE IS AT LEAST ONE ARC IN D WHOSE ORIGIN IS v, AND WHICH DOES NOT BELONG TO T.

THEREFORE T CAN BE EXTENDED BY ONE ARC. (i.e. SINCE $id(v) = od(v)$, IF YOU CAN GET INTO V YOU CAN GET OUT OF V.)

$\therefore$ T CAN BE EXTENDED TO A CIRCUIT C.

IF C CONTAINS ALL ARCS OF D WE ARE DONE. OTHERWISE LET $H_1, \ldots, H_k$ BE THE (WEAK) COMPONENTS OF $D - E(C)$.

EACH H_i HAS EQUAL INDEGREES AND OUTDEGREES (WHY?), AND FEWER ARCS THAN D, SO BY OUR INDUCTION HYPOTHESIS CONTAINS AN EULERIAN CIRCUIT C_i $(1 \le i \le k)$.

NOW BUILD AN EULERIAN CIRCUIT IN D BY TRAVERSING C UNTIL A NON-ISOLATED COMPONENT H_i OF $D - C$ IS REACHED. TRAVERSE C_i, THEN CONTINUE ALONG C TO THE NEXT NON-ISOLATED COMPONENT OF $D - C$. PROCEED IN THIS MANNER UNTIL THE CIRCUIT C IS COMPLETE.

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THEOREM

LET D BE A (WEAKLY) CONNECTED DIGRAPH WITH AT LEAST TWO VERTICES. THEN D IS SEMI-EULERIAN IF AND ONLY IF THERE EXIST TWO VERTICES u, v SUCH THAT

$$od(u) = id(u) + 1 \quad , \quad id(v) = od(v) + 1 \; ;$$

AND FOR ALL OTHER VERTICES w, $id(w) = od(w)$. (THE SEMI-EULERIAN TRAIL THEN HAS ORIGIN u AND TERMINUS v.)

EXERCISE

PROVE THIS OR READ THE PROOF OF THEOREM 7.3 ON P.241.