7.1 DIRECTED GRAPHS: DEFINITIONS

A DIRECTED GRAPH (OR DIGRAPH) CONSISTS OF A PAIR $D = (V, A)$ OF VERTICES $V = V(D)$ AND DIRECTED ARCS $A = A(D) \subseteq V \times V$.

i.e. DIRECTED ARCS ARE ORDERED PAIRS OF VERTICES.

NOTE. $A(D)$ CAN BE THOUGHT OF AS A RELATION ON $V(D)$! uAv IFF $(u,v) \in A(D)$.

IF $a = (u,v) \in A$ WE SAY a IS AN ARC FROM u TO v, u IS THE ORIGIN OF a, AND v IS THE TERMINUS OF a.

GIVEN A DIRECTED GRAPH D, THE UNDERLYING (UNDIRECTED) GRAPH G IS OBTAINED BY REPLACING EACH DIRECTED ARC (ORDERED PAIR) WITH AN EDGE HAVING THE SAME ENDS (UNORDERED PAIR.)

GIVEN AN UNDIRECTED GRAPH G, AN ORIENTATION OF G IS OBTAINED BY REPLACING EACH EDGE BY A DIRECTED ARC. NOTE G HAS MANY DIFFERENT ORIENTATIONS.

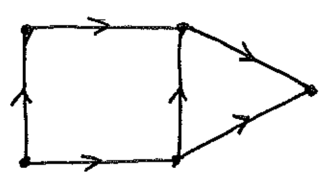
WE CAN AND DO SPEAK OF THE DIRECTED ANALOGUES OF : WALK, TRAIL, PATH, CYCLE, ETC.. WITH OBVIOUS DEFINITIONS.

WE SAY $v \in V(D)$ IS REACHABLE FROM $u \in V(D)$ IF THERE EXISTS A DIRECTED u-v PATH IN D. WE CALL u THE ORIGIN AND v THE TERMINUS OF SUCH A PATH.

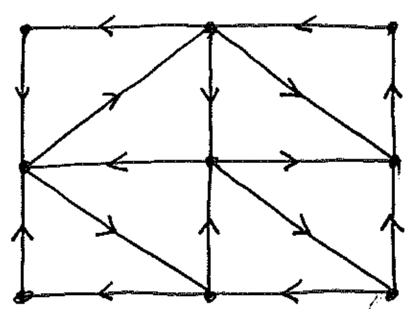
D IS CALLED (WEAKLY) CONNECTED IF ITS UNDERLYING GRAPH IS CONNECTED.

D IS CALLED STRONGLY CONNECTED IF EVERY VERTEX IN D IS REACHABLE FROM EVERY OTHER VERTEX.

EX. CONNECTED BUT NOT STRONGLY CONNECTED.



EX. STRONGLY CONNECTED.

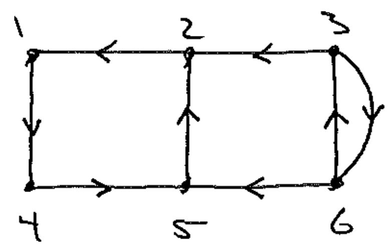


A <u>sub-digraph</u> of $D=(V,A)$ is a digraph $D'=(V',A')$ where $V'\subseteq V$ and $A'\subseteq A$.

A <u>strong component</u> of D is a sub-digraph S which is strongly connected, and not a proper sub-digraph of any other strongly connected sub-digraph of D.

i.e. S is maximal with respect to the property of being strongly connected.

<u>Ex.</u>



The strong components in this example are the subdigraphs induced by $\{3,6\}$ and $\{1,2,4,5\}$.

D is called simple if it has no parallel arcs or directed loops:

Note  is not an example of parallel arcs.

TWO DIGRAPHS D_1, D_2 ARE CALLED <u>ISOMORPHIC</u> IF THERE ARE BIJECTIONS

$$f : V(D_1) \to V(D_2)$$
$$g : A(D_1) \to A(D_2)$$

SUCH THAT FOR ALL DIRECTED ARCS $(u,v) \in A(D_1)$:

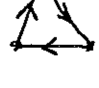
$$g(u,v) = (f(u), f(v)) \in A(D_2) \, .$$

IT FOLLOWS THAT THE SAME PROPERTY HOLDS FOR f^{-1} AND g^{-1}: FOR $(x,y) \in A(D_2)$:

$$g^{-1}(x,y) = (f^{-1}(x), f^{-1}(y)) \, .$$

WE WRITE $D_1 \cong D_2$ TO SAY THAT D_1 IS ISOMORPHIC TO D_2.

<u>EX.</u> K_3  HAS 8 ORIENTATIONS:

D_1  D_5 

D_2  D_6 

D_3  D_7 

D_4  D_8  .

OBSERVE THAT $D_1 \cong D_8$ AND $D_2 \cong D_3 \cong \ldots \cong D_7$.

EXERCISE

HOW MANY ORIENTATIONS HAS K_n? $2^{\binom{n}{2}}$.

EXERCISE (HARD)

UP TO ISOMORPHISM, HOW MANY DIGRAPHS HAVE K_n AS THEIR UNDERLYING GRAPH.

LET $V(D) = \{v_1, \ldots, v_n\}$, $A(D) = \{a_1, \ldots, a_m\}$. THE ADJACENCY MATRIX $A = (a_{ij})$ OF D IS THE $n \times n$ MATRIX WITH

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in A(D) \\ 0 & \text{otherwise} \end{cases}$$

NOTE: WE RESTRICT TO SIMPLE DIGRAPHS HERE ALTHOUGH A GENERAL DEFINITION IS POSSIBLE.

OBSERVE THAT A IS JUST THE MATRIX REPRESENTATION OF THE RELATION $A(D)$ ON V. THUS IT IS AN APPROPRIATE USE (MISUSE?) OF NOTATION TO WRITE $A(D)$ FOR BOTH THE ARC SET AND THE ADJACENCY MATRIX.

NOTE $A(D)$ NEED NOT BE SYMMETRIC.

THE INCIDENCE MATRIX $M = M(D)$ IS THE $n \times m$ MATRIX $M = (m_{ij})$ WITH

$$m_{ij} = \begin{cases} 0 & \text{IF } v_i, a_j \text{ NOT INCIDENT} \\ -1 & \text{IF } v_i \text{ IS ORIGIN OF } a_j \\ 1 & \text{IF } v_i \text{ IS TERMINUS OF } a_j \end{cases} .$$

OBSERVE THAT $A(D)$ DEPENDS ON THE VERTEX LABELS, AND $M(D)$ DEPENDS ON BOTH VERTEX AND ARC LABELS.