CMPE 177 7-8-09

THM Let T BE A GRAPH WITH n VERTICES AND m EDGES. T.F.A.E.

(1) T is A TREE (CONN. & Acyclic).

(2) T is Acyclic & $m = n-1$.

(3) T is CONNECTED & $m = n-1$.

(4) T is CONNECTED & EACH EDGE is A BRIDGE.

(5) ANY TWO VERTICES OF T ARE CONNECTED BY A UNIQUE PATH.

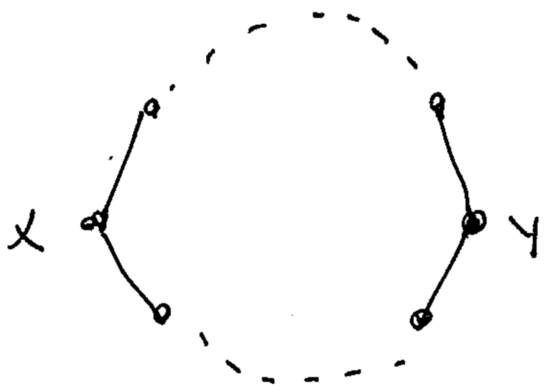
(6) T is Acyclic, BUT ADDITION OF ANY EDGE CREATES A UNIQUE CYCLE.

WE SHOWED $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$

MUST SHOW $5 \Rightarrow 6 \Rightarrow 1$.

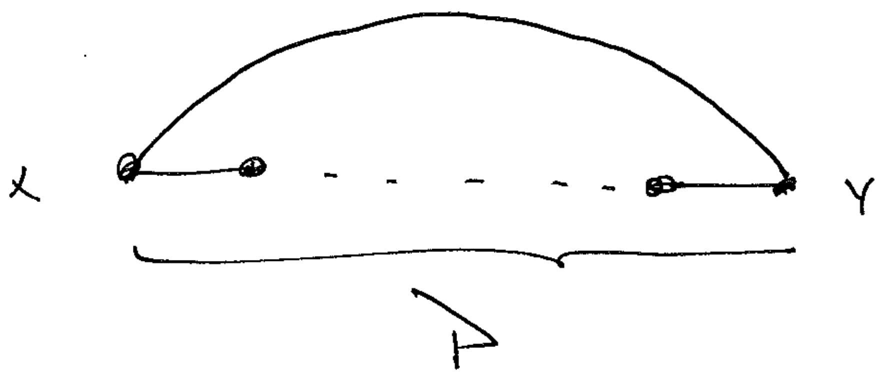
$\boxed{(5) \Rightarrow (6)}$.

SUPPOSE EACH PAIR OF VERTICES IN T ARE JOINED BY A UNIQUE PATH. SUPPOSE, TO GET A $\times$, THAT T CONTAINS A CYCLE.



THEN ANY TWO VERTICES ON THAT CYCLE ARE JOINED BY DISTINCT PATHS, CONTRARY TO HYPOTHESIS. $\therefore$ T IS ACYCLIC.

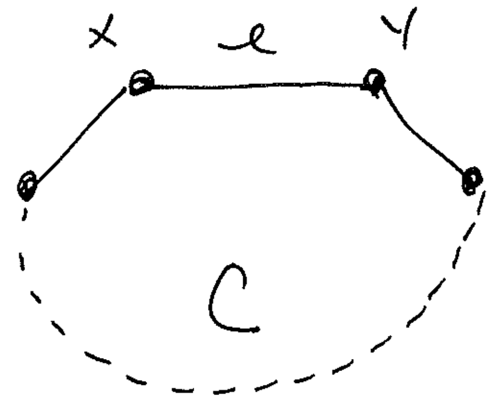
LET $x, y \in V(T)$. SAY $x \neq y$ (CASE $x = y$ IS TRIVIAL). IF WE JOIN x TO y WITH A NEW EDGE, THAT EDGE TOGETHER WITH THE UNIQUE PATH P IN T JOINING x TO y, FORMS A CYCLE.



If more than one cycle were created, then x, y must have been joined by distinct paths to begin with. ///

$\boxed{(6) \Rightarrow (1)}$.

Suppose T is acyclic, but addition of any edge creates a unique cycle. Must show T is conn. Pick any $x, y \in V(T)$. Say x, y non-adjacent, and $x \neq y$. Join x to y by a new edge e, which creates a single cycle C in $T + e$.



$\therefore$ $C - e$ is an x-y path in T.

$\therefore$ x is reachable from y in T,

$\therefore$ T is connected. ///.

<u>Corollary</u>

A forest G with n vertices and k conn. components must have $n-k$ edges.

<u>Proof:</u>

Say the k components of G are

$$T_1, T_2, \ldots, T_k$$

each of which are trees.

LET $n_i = |V(T_i)|$ $(1 \le i \le k)$

$m_i = |E(T_i)|$ $(1 \le i \le k)$

BY EITHER (2) OR (3) OF LAST THM WE HAVE $m_i = n_i - 1$

$$\therefore |E(G)| = \sum_{i=1}^{k} m_i$$

$$= \sum_{i=1}^{k} (n_i - 1)$$

$$= \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$$

$$= n - k. \qquad ///$$

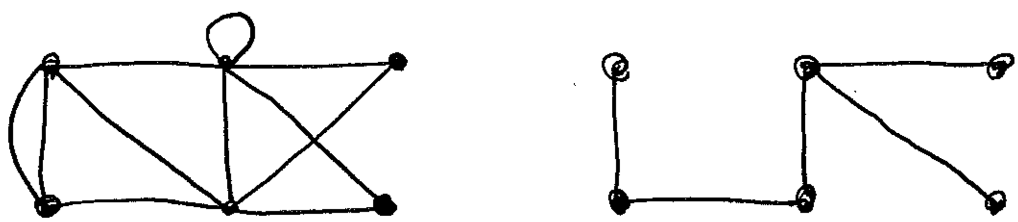
<u>EXERCISE</u>: PROVE THE CONVERSE.

i.e. A GRAPH ON n VERTICES, k COMPONENTS, AND $m = n-k$ EDGES IS NECESSARILY Acyclic.

(2.3) SPANNING TREES

<u>Defn</u>. A <u>SPANNING TREE</u> IN G IS A SUB-TREE WHICH INCLUDES ALL VERTICES.

<u>EX.</u>



G T

<u>THM</u> G CONTAINS A SPANNING TREE IFF G IS CONNECTED.

<u>PROOF</u>:

($\Rightarrow$) SUPPOSE G CONTAINS A SPANNING TREE. THEN ANY TWO VERTICES IN G ARE JOINED BY A PATH CONSISTING OF TREE EDGES. $\therefore$ G IS CONNECTED.

($\Leftarrow$) SUPPOSE G IS CONNECTED. A SP. TREE CAN BE CONSTRUCTED BY THE FOLLOWING ALGORITHM.

TOP DOWN

1.) while THERE EXISTS A CYCLE IN G
2.) SELECT SUCH A CYCLE AND REMOVE ONE OF ITS EDGES.

THE RESULTING SUBGRAPH T INCLUDES ALL VERTICES, IS CONN., AND IS ACYCLIC, $\therefore$ T IS A SPANNING TREE. ///

ANOTHER ALGORITHM.

BOTTOM UP

1.) LET T CONSIST OF A SINGLE VERTEX IN G
2.) while T DOES NOT INCLUDE ALL VERTICES.
3.) FIND AN EDGE WITH ONE END IN T, AND ONE END NOT IN T, ADD THAT EDGE AND ITS 'OTHER' END TO T.

NOTE: SUCH AN EDGE EXISTS SINCE G IS CONNECTED.

EXERCISE: PROVE THIS, i.e. SHOW THAT IF T DOES NOT INCLUDE ALL VERTICES, THEN THERE EXISTS AN EDGE $e = xy$ s.t. $x \in V(T)$ AND $y \notin V(T)$.

EXERCISE SHOW THAT THE SUBGRAPH T OBTAINED 'BOTTOM-UP' IS A SP. TREE IN G (USE PART (2) OF TREENESS THM.)

HOW MANY SP. TREES DOES A CONN. GRAPH HAVE.

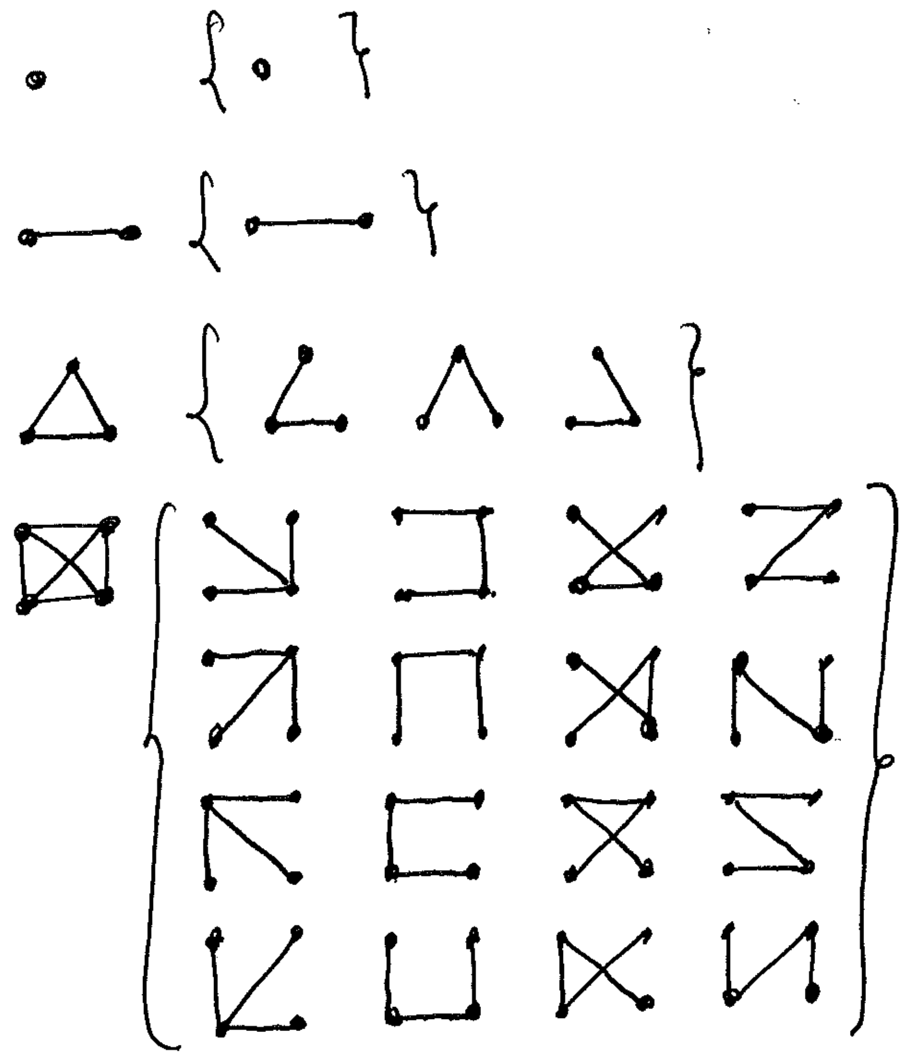
LET G BE A SIMPLE CONNECTED GRAPH.

Define

$$S(G) = \{\text{SPANNING TREES IN } G\}.$$

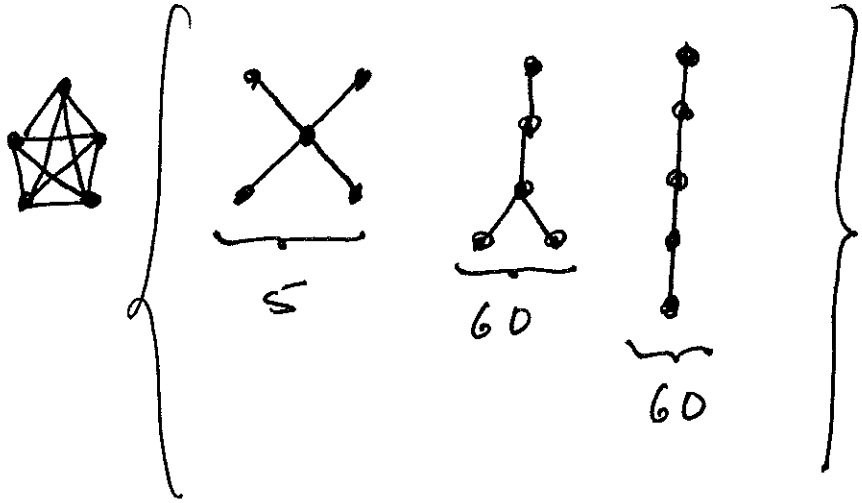
<u>GOAL</u>: DEVELOP A FORMULA FOR $|S(G)|$.

CONSIDER COMPLETE GRAPHS $G = K_n$ AS EXAMPLES.



	#
K_1	$1 = 1^{1-2}$
K_2	$1 = 2^{2-2}$
K_3	$3 = 3^{3-2}$
K_4	$16 = 4^{4-2}$

K_5



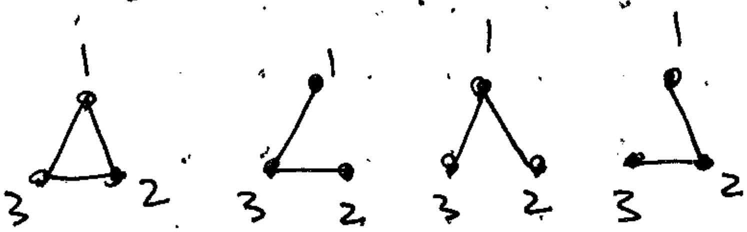
$125 = 5^{5-2}$

THM: $|S(K_n)| = n^{n-2}$

(CAYLEYS THEOREM.)

NOTE: THE # OF SP. TREES IN K_n EQUALS THE THE # OF LABELED TREES ON n VERTICES

K_3



WE WILL FIND A GENERAL FORMULA FOR $|S(G)|$ FOR ANY CONN. SIMPLE GRAPH G.

LET $V(G)=\{v_1,\dots,v_n\}$ AND

$E(G)=\{e_1,\dots,e_m\}$

(WE KNOW THAT $n-1 \le m \le \frac{n(n-1)}{2}$

LET $M=M(G)$ BE INCIDENCE MATRIX.

SINCE G IS SIMPLE M IS A 0-1 MATRIX WITH 2 1'S IN EACH COLUMN.

ARBITRARILY REPLACE ONE 1 IN EACH COLUMN BY -1.

$$\begin{array}{cc} k & \\ \vec{M} = \begin{pmatrix} \vdots \\ -1 \\ \vdots \\ +1 \\ \vdots \end{pmatrix} & \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \end{array} \qquad v_i \xrightarrow{e_k} v_j$$

ORIENTED INCIDENCE MATRIX.

NOTE: if you add all rows of $\vec{M}$ you get the 0 vector.

Defn The Laplacian of G is the $n \times n$ symmetric matrix

$$L = L(G) = \underset{n \times m}{\vec{M}} \; \underset{m \times n}{\vec{M}^T}$$

NOTE: $L^T = (\vec{M}\vec{M}^T)^T = \vec{M}^{TT}\vec{M}^T = \vec{M}\vec{M}^T = L$

Let L_{rr} denote the r-r cofactor of L, i.e. the matrix obtained by deleting the r^{th} row & r^{th} column from L. (for r in the range $1 \le r \le n$.)

NOTE: L_{rr} is of size $(n-1) \times (n-1)$

MATRIX TREE THEOREM (KIRCHOFF)

FOR ANY r $(1 \le r \le n)$:

$$|S(G)| = \det(L_{rr}).$$

BEFORE WE PROVE THIS, NOTE THERE IS AN ALTERNATE DEFINITION OF L.

LET $D = D(G)$ BE THE DEGREE MATRIX OF G, i.e. 0's OFF THE MAIN DIAGONAL, AND THE ii^{th} ENTRY IS DEGREE OF v_i. ALSO LET $A = A(G)$ BE THE ADJ. MATRIX.

THEN

$$L = D - A.$$

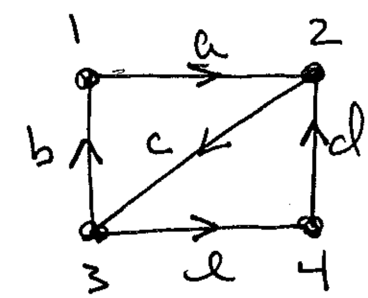
EXERCISE:

SHOW THAT THESE DEFINITIONS OF L ARE EQUIVALENT, i.e. SHOW

$$\vec{M}\vec{M}^T = D - A$$

OBSERVE: RHS IS INDEPENDENT OF EDGE LABELS & ORIENTATIONS, HENCE SO IS LHS.

EX.



$$\vec{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \\ \begin{array}{ccccc} a & b & c & d & e \end{array} \end{array}$$

$$L = \tilde{M}\vec{M}^T = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Pick $r = 2$. Then

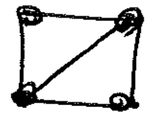
$$L_{22} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

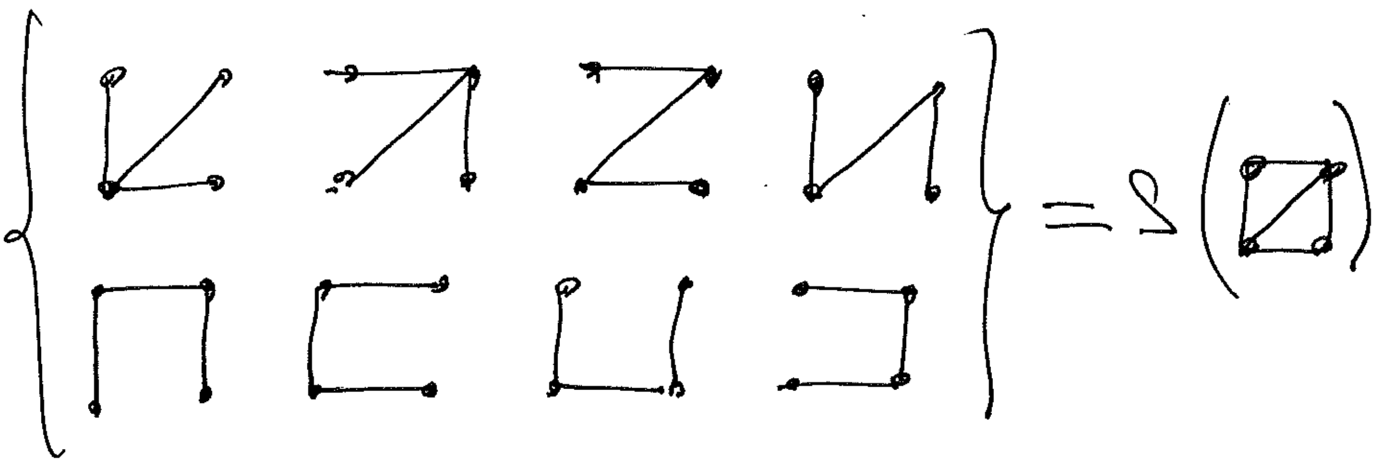
$$\det(L_{22}) = 2\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} & \end{vmatrix}$$

$$= 2(6-1) + (-2-0) + 0$$

$$= 10 - 2$$

$$= 8$$

THE 8 SP. TREES OF  ARE



A FACT FROM LINEAR ALGEBRA USED IN THE PROOF:

LET P BE AN $s \times t$ MATRIX, AND Q A $t \times s$ MATRIX (WHERE $s \leq t$). THEN PQ IS AN $s \times s$ SQUARE MATRIX.

LET $I = \{i_1, i_2, \ldots, i_s\}$ WITH $1 \leq i_1 < i_2 < i_3 < \ldots < i_s \leq t$.

LET P_I = $s \times s$ SUBMATRIX OF P WITH COLUMNS INDEXED BY I

Q^I = $s \times s$ SUBMATRIX OF Q WITH ROWS INDEXED BY I.

Thm (CAUCHY-BINET)

$$\det(PQ) = \sum_I \det(P_I) \cdot \det(Q^I)$$