CMPE 177 7-6-09

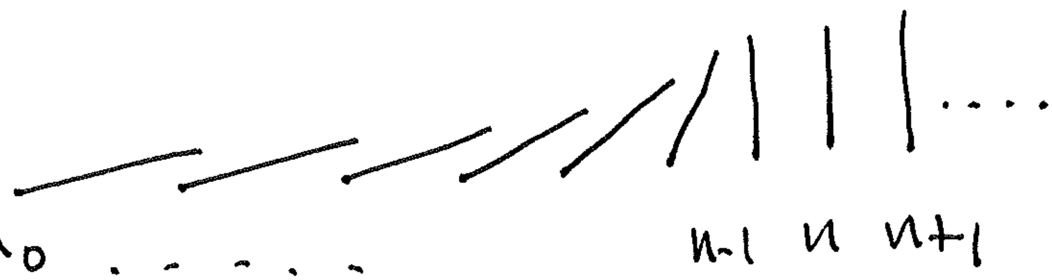
- mid 1. Wed July 15
- Possible Rev. Sess. Tue July 14
  when?. how long?
  5-7.
- hw2: will post solns tonight.

Induction Proofs:

Let $P(n)$ stand for a propositional function. Wish to prove:

$$\forall n \geq n_0 : P(n)$$

Domino Analogy:



$P(n) =$ 'the n^{th} domino falls'.

Proof by mathematical induction:

2 steps

BASE

I. show directly $P(n_0)$ is true.

INDUCTION

IIa. show: for any $n \geq n_0$: if $P(n)$, then $P(n+1)$ is true.

Conclude $\forall n \geq n_0$: $P(n)$.

IND. HYPOTHESIS

IIb. show: for any $n > n_0$: if $P(n-1)$ is true, then $P(n)$ is true.

IIa, IIb are called <u>WEAK INDUCTION</u>

<u>STRONG INDUCTION</u>

IIc. show: for any $n \geq n_0$, If

$\underbrace{P(n_0), P(n_0+1), \ldots, P(n) \text{ ARE TRUE}}_{\text{IND. HYP.}}$,

THEN $P(n+1)$ is TRUE.

IId. show: for any $n > n_0$, If

$\underbrace{P(n_0), P(n_0+1), \ldots\ldots, P(n-1) \text{ ARE TRUE}}_{\text{IND. HYP.}}$,

THEN $P(n)$ is TRUE

EX. (WEAK INDUCTION II a).

FOR ALL $n \geq 1$: $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}} \Leftarrow P(n)$

PROOF:

I. $P(1)$ says $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$, i.e.

$1 = \frac{1 \cdot 2}{2}$ i.e. $1 = 1$, which is true.

IIa. show: for any $n \geq 1$: $P(n) \Rightarrow P(n+1)$.

LET $n \geq 1$. ASSUME $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

MUST SHOW: $\sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$.

SO $$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + (n+1)$$

$$= \left(\frac{n(n+1)}{2}\right) + (n+1) \quad \left\{ \begin{array}{l} \text{BY IND.} \\ \text{HYP.} \end{array} \right.$$

$$= (n+1)\left[\frac{n}{2} + \frac{2}{2}\right] = \frac{(n+1)(n+2)}{2}. \quad ///.$$

Lemma

Let G be a graph with n vertices, m edges, and k conn. components.
Then: $m \geq n - k$, i.e.

$$|E(G)| \geq |V(G)| - \omega(G) \qquad P(m)$$

Proof Induction on m. (strong Ind.)

I. Base case: $m = 0$. In this case all vertices are isolated. So each vertex is a conn. comp, i.e. $n = k$. So inequality becomes $0 \geq 0$, which is true.

II d. show for any $m > 0$: If $P(0), P(1), \ldots, P(m-1)$ are true, then $P(m)$ is true.

LET $m > 0$. ASSUME FOR ANY GRAPH G' WITH $|E(G')| < m$, THAT $|E(G')| \geq |V(G')| - \omega(G')$.

LET G HAVE n VERTICES, m EDGES, AND k COMPONENTS. MUST SHOW THAT $m \geq n - k$.

PICK ANY $e \in E(G)$. WE APPLY IND. HYP. TO $G-e$, NOTE.

$$|E(G-e)| = m-1 < m$$

RECALL, BY AN EARLIER lemma, THAT EITHER

$$\text{①}\quad \omega(G-e) = \omega(G) = k$$

or

$$\text{②}\quad \omega(G-e) = \omega(G)+1 = k+1$$

(e is a BRIDGE)

CASE ①: BY THE IND HYPOTHESIS

$$|E(G-e)| \geq |V(G-e)| - \omega(G-e)$$

i.e. $m-1 \geq n-k$

$\therefore\ m \geq n-k+1 > n-k$

$\therefore\ m \geq n-k$. ✓

CASE (2) $\omega(G-e) = k+1$.

By Ind. Hyp.

$$E(G-e) \geq |V(G-e)| - \omega(G-e)$$

$$\therefore \quad m-1 \geq n-(k+1)$$

$$\therefore \quad m-\cancel{1} \geq n-k-\cancel{1}$$

$$\therefore \quad m \geq n-k \quad \checkmark$$

Recall: A tree T is a graph which is both

(a) connected, and

(b) acyclic.

THEOREM (TREENESS).

LET T BE A GRAPH ON n VERTICES. THEN THE FOLLOWING ARE EQUIVALENT.

(1) T IS A TREE (i.e. CONN & ACYCLIC).

(2) T IS ACYCLIC, AND $|E(T)| = n-1$.

(3) T IS CONNECTED, AND $|E(T)| = n-1$.

(4) T IS CONNECTED, AND EACH EDGE IS A BRIDGE.

(5) ANY TWO VERTICES OF T ARE JOINED BY A UNIQUE PATH.

(6) T IS ACYCLIC, BUT ADDITION OF ANY NEW EDGE CREATES A UNIQUE CYCLE.

EXAMPLES

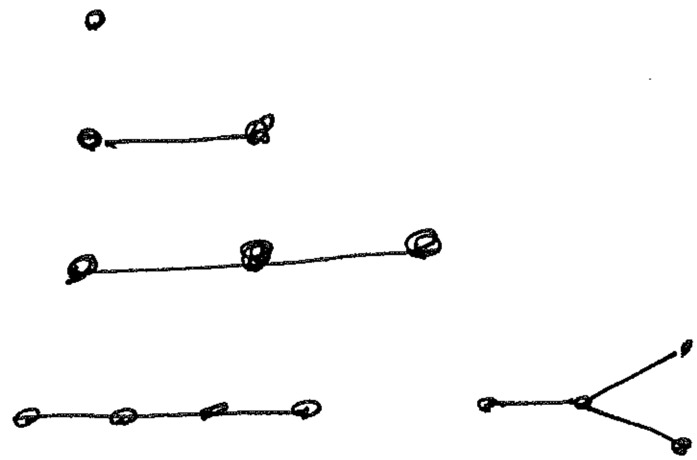
$m = \#$ EDGES.

$n = 1$: $m = 0$

$n = 2$: $m = 1$

$n = 3$: $m = 2$

$n = 4$: $m = 3$



PROOF:

$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 1$.

$\boxed{1 \Rightarrow 2}$ INDUCTION ON $m = \#$ EDGES.

ANY OF ABOVE EXAMPLES CAN BE TAKEN AS BASE STEP.

LET $m > 0$, AND ASSUME FOR ANY TREE T' WITH $|E(T')| < m$, THAT $|E(T')| = |V(T')| - 1$.

LET T BE A TREE WITH m EDGES, AND n VERTICES. MUST SHOW $m = n - 1$.

PICK ANY $e \in E(T)$, AND REMOVE IT. SINCE T CONTAINS NO CYCLES, $T-e$ IS DISCONNECTED, IN FACT $T-e$ CONSISTS OF TWO SUB-TREES T_1, T_2 EACH WITH FEWER EDGES THAN T. i.e. $|E(T_i)| < m$ $(i=1,2)$.

BY IND. HYP.

$$|E(T_i)| = |V(T_i)| - 1 \qquad (i=1,2).$$

BUT ALSO $|V(T_1)| + |V(T_2)| = |V(T)| = n$, SINCE NO VERTICES WERE REMOVED.

SO

$$\begin{aligned} m = |E(T)| &= |E(T_1)| + |E(T_2)| + 1 \\ &= (|V(T_1)| - 1) + (|V(T_2)| \cancel{-1}) \cancel{+1} \\ &= |V(T_1)| + |V(T_2)| - 1 = n-1. \quad /// \end{aligned}$$

$\boxed{(2) \Rightarrow (3)}$ If T is acyclic & $m = n-1$, then T is connected.

Let T be an acyclic graph with $m = n-1$ edges. Let $k = \omega(T)$. Must show $k = 1$, for then T is connected. Each component of T is a tree, call them $T_1, T_2, \ldots, T_k$. Apply last part to each T_i to get

$$|E(T_i)| = |V(T_i)| - 1 .$$

Also $n = \sum_{i=1}^{k} |V(T_i)|$. Also

$$n-1 = m = \sum_{i=1}^{k} |E(T_i)| = \sum_{i=1}^{k} \left(|V(T_i)| - 1 \right)$$

$$= \sum_{i=1}^{k} |V(T_i)| - k$$

$$= n - k$$

$\therefore$ $n-1 = n-k$ $\therefore$ $k = 1$.

$\therefore$ T is conn. ///.

$\boxed{(3) \Rightarrow (4)}$

Show: If T is conn and $m = n-1$, then each edge of T is a bridge. Recall that for any graph on n vertices, m edges, & k components that $m \geq n-k$.

Pick any $e \in E(T)$ and apply this to $T - e$. By above fact

$$E(T-e) \geq |V(T-e)| - \omega(T-e)$$

$$\therefore \quad n-2 \geq n - \omega(T-e)$$

$$\therefore \quad \omega(T-e) \geq 2$$

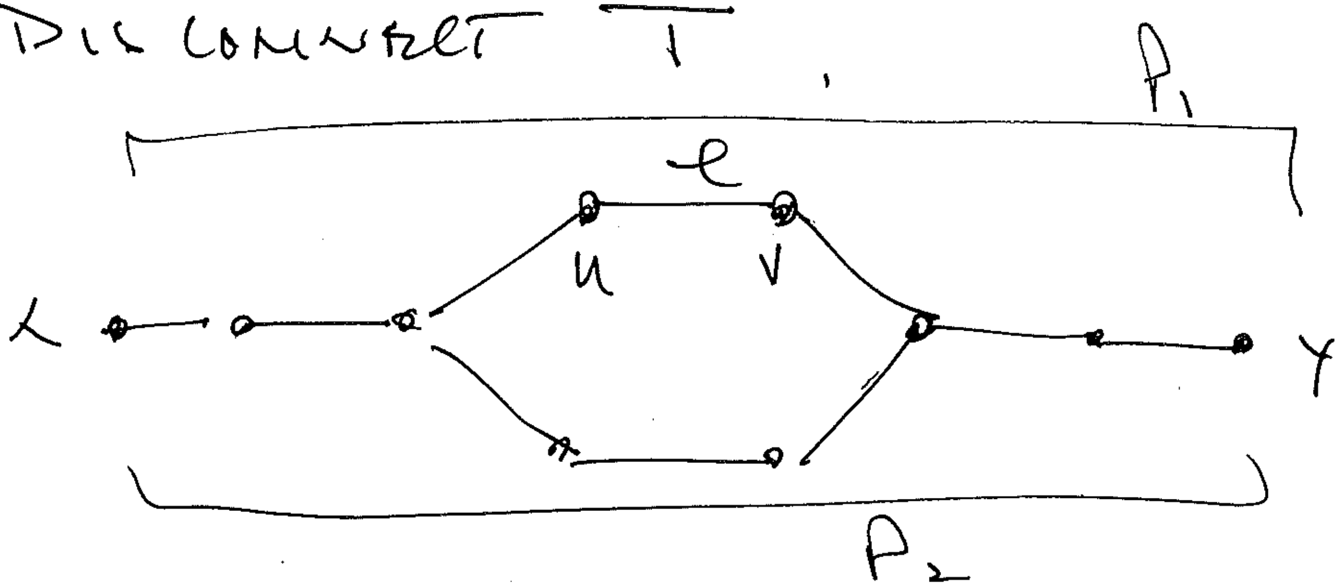
$\therefore$ $T-e$ IS DISCONNECTED.

$\therefore$ e IS A BRIDGE.

SINCE $e \in E(T)$ WAS CHOSEN ARBITRARILY, EVERY EDGE OF T IS A BRIDGE. ///

$\boxed{(4) \Rightarrow (5)}$. IF T IS CONNECTED AND ANY EDGE IS A BRIDGE, THEN ANY TWO VERTICES X, Y ARE JOINED BY A UNIQUE PATH. KNOW ANY $X, Y \in V(T)$ ARE JOINED BY AT LEAST ONE PATH, SINCE T IS CONN.

SUPPOSE T CONTAINS TWO X-Y PATHS P_1 & P_2, AND $P_1 \neq P_2$. THEN P_1, P_2 CANNOT HAVE ALL COMMON EDGES. SO THERE MUST BE SOME EDGE IN P_1 NOT IN P_2. CALL IT e. SO REMOVAL OF e DOES NOT DISCONNECT T.



THIS IS BECAUSE

T - e STILL CONTAINS A u-v PATH

(WHERE $e = uv$) BY SPLICING PARTS OF P_1 & P_2.

THIS CONTRADICTS THAT EVERY EDGE IS A BRIDGE. SO NO TWO SUCH PATHS P_1 & P_2 CAN EXIST.

$\therefore$ FOR ANY $x, y \in V(T)$, T CONTAINS A UNIQUE x-y PATH.

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