CMPE 177 7-27-09

Recall:

Defn: Let G be a conn. graph. $W \subseteq V(G)$ is called a separating set iff $G - W$ is disconn.

K_n has no separating set. Also

Exercise: if $G \not\cong K_n$ then G does contain a separating set. ($|V(G)| \geq 3$)

Defn: Let G be simple $G \not\cong K_n$. The (vertex) connectivity of G is

$$\kappa(G) = \min\{|W| : W \subseteq V(G) \text{ is a sep. set}\}.$$

DEFINE $\kappa(K_n) = n-1$, i.e. THE # OF VERTICES RESULTS IN K_1.

<u>DEFN</u>: A SET $F \subseteq E(G)$ IS CALLED <u>DISCONNECTING</u> IFF $G-F$ IS DISCONNECTED.

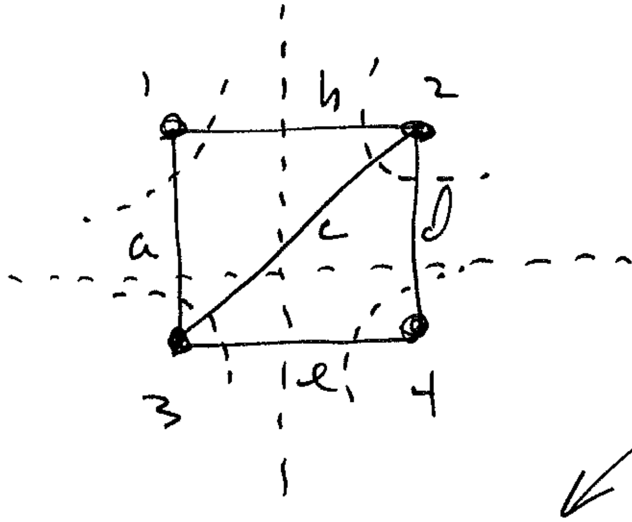
<u>DEFINE</u>: <u>EDGE CONNECTIVITY</u> OF G IS

$$\lambda(G) = \min\{|F| \mid F \text{ is a disconnecting set}\}$$

BY CONVENTION $\lambda(K_1) = 0$.

<u>DEFN</u>: A <u>CUTSET</u> IN G IS A DISCONN. SET $F \subseteq E(G)$ WHICH CONTAINS NO PROPER DISCONNECTING SUBSET.

E



$\{a,c\}$, $\{a,d\}$, $\{c,d\}$ — NOT DISCONNECTING SETS

CUTSETS: $\{a,c,d\}$, $\{a,b\}$, $\{e,d\}$

$\{a,c,e\}$, $\{b,c,d\}$, $\{b,c,e\}$

ALT DEFN: $\lambda(G)$ = size of a smallest CUTSET.

IN THIS EX. $\lambda(G) = 2$.

ALSO IN THIS EX. $\kappa(G) = 2$.

$\{2,3\}$ IS A SEP. SET & G CONTAINS NO CUT VERTEX.

<u>NOTATION</u>: $\delta(G)$ = MINIMUM VERTEX DEGREE IN G.

<u>THM</u>: (Whitney's inequality)

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

<u>PROOF LATER</u>:

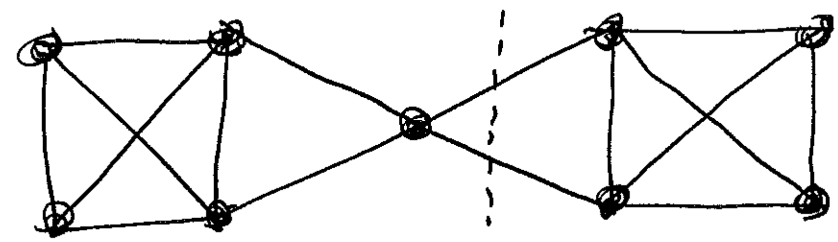
- NOTE $\kappa(G) \leq \delta(G)$ WAS A HW PROBLEM.
- Also $\lambda(G) \leq \delta(G)$ is NOT HARD.

<u>EXERCISE</u>.

<u>EX.</u> $G = K_n$.

$$\kappa(G) = \lambda(G) = \delta(G) = n-1$$

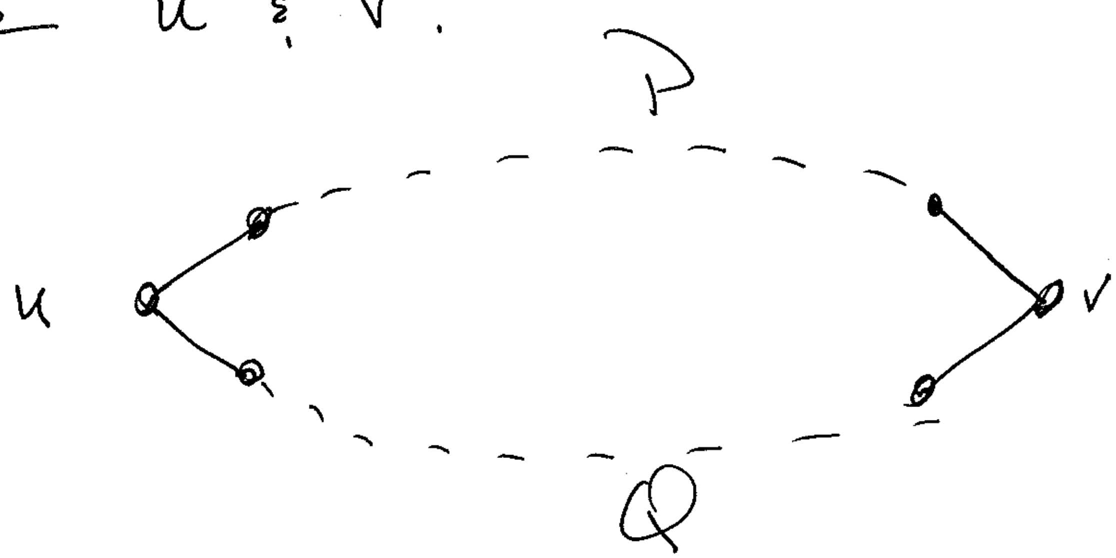
Ex.



$\kappa(G)=1,\ \lambda(G)=2,\ \delta(G)=3$

Defn

Let $u, v \in V(G)$. Let P, Q be two u-v paths in G. We say P, Q are internally disjoint iff their only common vertices are u & v.



Defn

We call G K-CONNECTED iff $K(G) \geq K$.

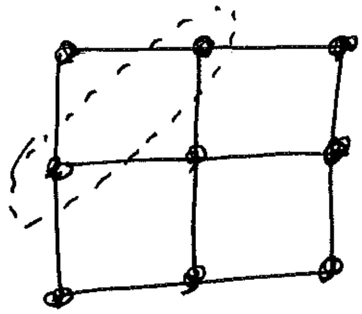
We call G K-EDGE CONNECTED iff $\lambda(G) \geq K$.

THM (WHITNEY)

Let G be a simple graph with at least $K+1$ vertices. Then G is K-connected iff for each pair of distinct vertices $u, v \in V(G)$, G contains a set of K u-v paths which are pairwise internally disjoint.

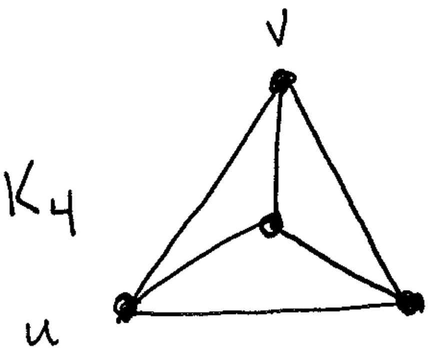
Proof later..

__Ex__

G 

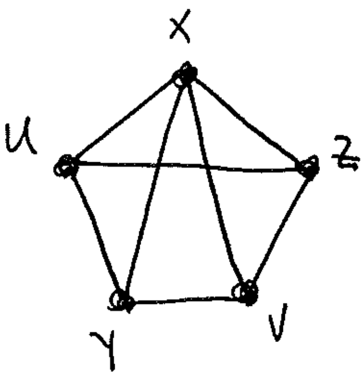
$\kappa(G) = 2$

__Ex.__

K_4 

$\kappa(K_4) = 3$

__Ex.__

G 

$\kappa(G) = 3$

$\{x, y, z\}$ is a sep. set

Result

NOTE: G is CONN. IFF G is 1-CONNECTED. A GRAPH WHICH is 2-CONNECTED is CALLED BICONNECTED.

MIDTERM MATERIAL STOPS HERE.

(3.1) EULERIAN GRAPHS

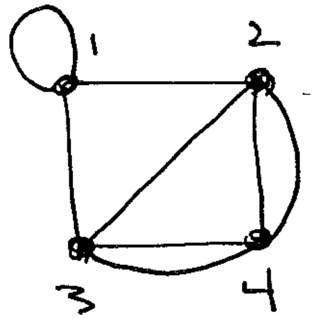
Recall Defns of walk, trail, path, cycle.

Defn An Euler trail is a trail in G which includes all edges.

An Euler tour is a closed Euler trail.

Defn G is Eulerian iff it contains an Euler tour. G is called semi-Eulerian iff it contains a non-closed Euler trail.

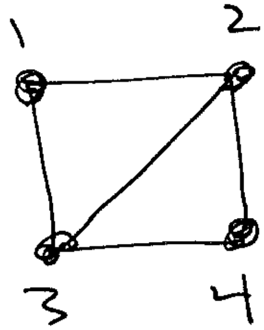
Ex.



EULER TOUR.

EULERIAN: 1, 2, 3, 4, 2, 4, 3, 1, 1

Ex

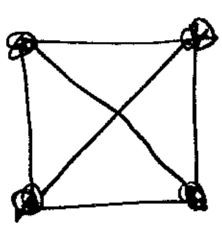


NON-CLOSED
EULER TRAIL.

SEMI-EULERIAN: 2, 1, 3, 2, 4, 3

Ex.

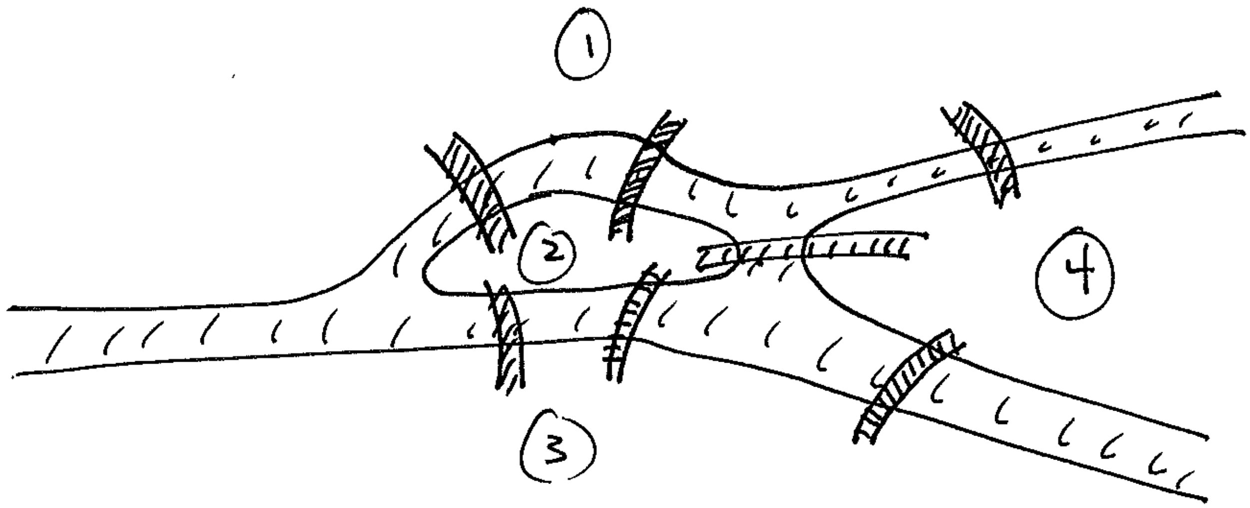
K_4



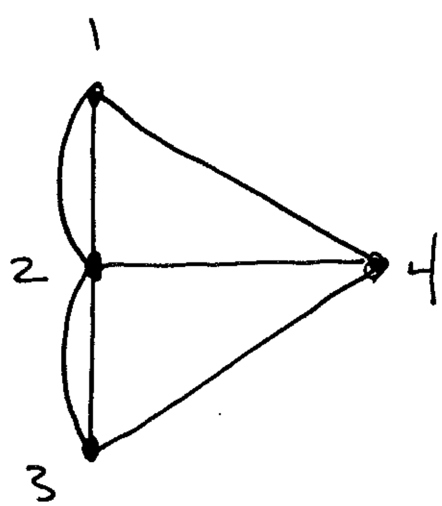
NON-EULERIAN

NON-SEMI EULERIAN.

KÖNIGSBERG BRIDGE PROBLEM



CAN YOU CROSS EACH BRIDGE EXACTLY ONCE.



IS THIS GRAPH EULERIAN OR SEMI-EULERIAN OR NEITHER?

THEOREM (EULER)

A CONN. GRAPH IS EULERIAN IFF EVERY VERTEX HAS EVEN DEGREE.

Lemma

If vertices of G all have degree at least 2 (i.e. if $\delta(G) \geq 2$) then G contains a cycle.

Proof:

Suppose G is simple (otherwise result is trivially true.) Let $x_0 \in V(G)$. Let x_1 be any neighbor of x. (x_1 exists since $\deg(x_0) \geq 2$.)

Let x_2 be any neighbor of x_1, other than x (x_2 exists since $\deg(x_1) \geq 2$. Continue in the same manner to find x_{i+1}, a neighbor of x_i other than x_{i-1} (for $i \geq 1$).

EXTEND THE PATH

$$x \to x_1 \to x_2 \cdots \to x_i \to$$

UNTIL SOME VERTEX x_k IS VISITED TWICE (MUST HAPPEN SINCE $V(G)$ IS FINITE.) THEN

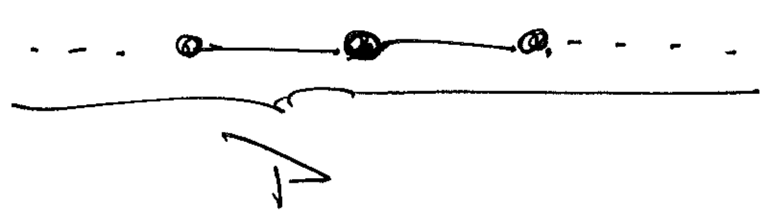
$$x_k \to x_{k+1} \to \cdots \to x_k$$

IS THE REQUIRED CYCLE IN G.

///.

<u>PROOF OF EULER'S THM</u>:

($\Rightarrow$) LET P BE AN EULER TOUR IN G. WHEN P PASSES THROUGH A VERTEX IT CONTRIBUTES TWO TO ITS DEGREE



SINCE EACH EDGE IS TRAVERSED EXACTLY ONCE, THE DEGREE OF EACH VERTEX IS THE SUM OF THESE CONTRIBUTIONS. ∴ ALL DEGREES ARE EVEN.

(⟸) SUPPOSE ALL VERTEX DEGREES ARE EVEN. MUST SHOW G CONTAINS AN EULER TOUR. WE USE INDUCTION ON # OF EDGES IN G.

<u>BASE CASE</u>: $|E(G)| = 0$. SINCE G IS CONN, IT CAN HAVE ONLY 1 VERTEX. INDEED 0 IS even, AND G CONTAINS AN EULER TOUR. (i.e. TRIVIAL PATH.)

ASSUME $|E(G)| > 0$, AND ASSUME AS IND. HYP. FOR ANY CONN. GRAPH H WITH $0 \le |E(H)| < |E(G)|$ THAT IF ALL DEGREES IN H ARE EVEN, THEN H CONTAINS AN EULER TOUR. SUPPOSE ALL DEGREES IN G ARE EVEN. SINCE G IS CONNECTED, IT HAS NO ISOLATED VERTICES, SO FOR ALL $x \in V(G)$, $\deg(x) \ge 2$. BY LAST LEMMA G CONTAINS A CYCLE, CALL IT C. LET H BE THE SPANNING SUBGRAPH OF G OBTAINED BY REMOVING THE

EDGES OF C. i.e.

$$V(H) = V(G)$$

$$E(H) = E(G) - E(C)$$

H may BE DISCONNECTED. LET $H_1, H_2, \ldots, H_k$ BE THE CONN. COMPONENTS OF H. i.e.

$$E(H) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$$

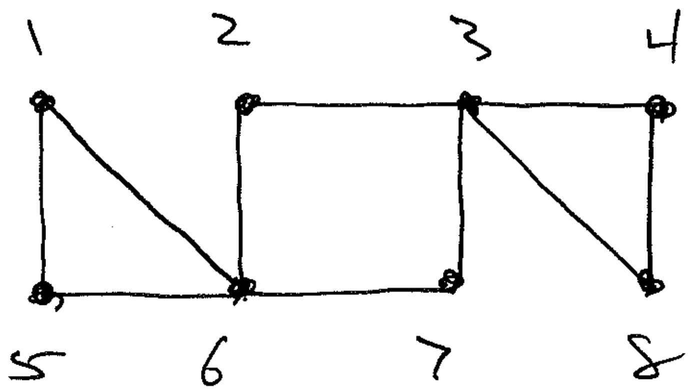
NOW EACH H_i is CONN. AND HAS FEWER EDGES THAN G.

ALSO EACH H_i HAS ALL EVEN VERTEX DEGREES. (why? REMOVING EDGES OF C DECREASES EVERY DEGREE BY EITHER 0 OR 2.)

SO BY IND. HYP. EACH H_i CONTAINS AN EULER TOUR.

EX.

G

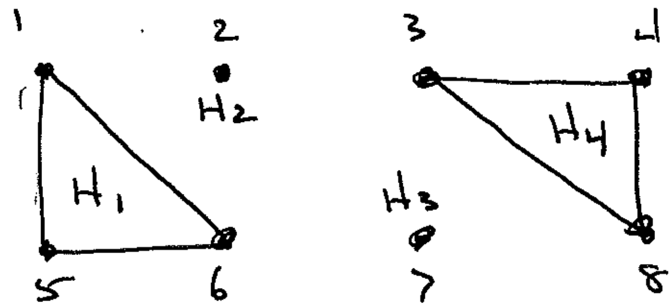


EULER TOUR

6, 5, 1, 6, 2, 3, 4, 8, 3, 7, 6

C: 2, 3, 7, 6, 2

H



CONTINUE ~~

WE CONSTRUCT AN EULER TOUR IN G AS FOLLOWS. START ON ANY VERTEX OF C. FOLLOW THE EDGES OF C UNTIL A NON-ISOLATED

VERTEX OF H IS REACHED, SAY IN COMPONENT H_i. FOLLOW THE EULER TOUR IN H_i THAT EXISTS BY IND. HYP. CONTINUE TO FOLLOW C UNTIL THE NEXT NON-ISOLATED VERTEX OF C IS ENCOUNTERED. CONTINUE IN THIS MANNER UNTIL YOU REACH INITIAL VERTEX ON C.

WE'VE TRAVERSED ALL EDGES EXACTLY ONCE SINCE

$$E(G) = E(C) \cup E(H_1) \cup \cdots \cup E(H_k)$$

$\therefore$ G IS EULERIAN.

COROLLARY

A CONN. GRAPH G IS EULERIAN IFF $E(G)$ CAN BE PARTITIONED INTO DISJOINT CYCLES.

PROOF:

BY LAST THM, WE MAY SHOW EQUIVALENTLY THAT: $E(G)$ CAN BE PARTIONED INTO DISJOINT CYCLES IFF EACH VERTEX DEGREE IS EVEN.

($\Rightarrow$) SUPPOSE $E(G)$ PARTITIONS INTO DISJOINT CYCLES. EACH CONTRIBUTES 2 TO THE DEGREES

OF EACH OF ITS VERTICES.
SINCE EACH EDGE OF G
BELONGS TO EXACTLY ON SUCH
CYCLE, THE DEGREE OF
EACH VERTEX IS THE SUM
OF THESE CONTRIBUTIONS,
HENCE IS EVEN.

($\Leftarrow$) NEXT TIME.