CMPE 177 7-22-09

LET $G=(V,E)$ BE A WEIGHTED GRAPH, i.e. HAVE A weight FCN.

$$w: E \to \mathbb{R}$$

Fix $s \in V$ (SOURCE). FOR ANY $v \in V$ DEFINE

$$\delta(s,v) = \begin{cases} \min\{w(P) : P \text{ is AN } s\text{-}v \text{ PATH}\} \\ \infty \text{ if no such PATH EXISTS} \end{cases}$$

SSSP Problem: DETERMINE $\delta(s,v)$ for all $v \in C(s)$, AND DETERMINE SHORTEST PATHS.

Dijkstra's Algorithm: maintains a label $\lambda(v)$, POSSIBLY INFINITE, WHICH, BY COMPLETION: $\lambda(v) = \delta(s,v)$

NOTE: Dijkstra REQUIRES $w(e) \geq 0$.

DIJKSTRA

```
λ(s) ← 0
for all v ∈ V − {s}
    λ(v) ← ∞
T ← V
while T ≠ ∅
    FIND u ∈ T s.t. λ(u) is MINIMUM.
    for all v ∈ T ADJACENT TO u
        λ(v) = min(λ(v), λ(u) + w(uv))
    T ← T − {u}
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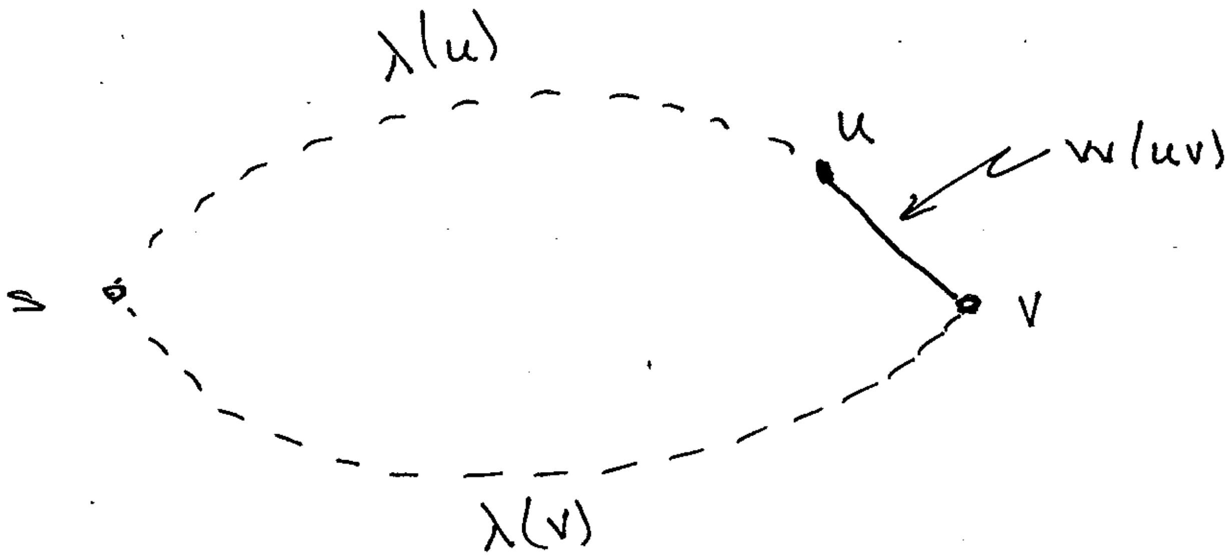
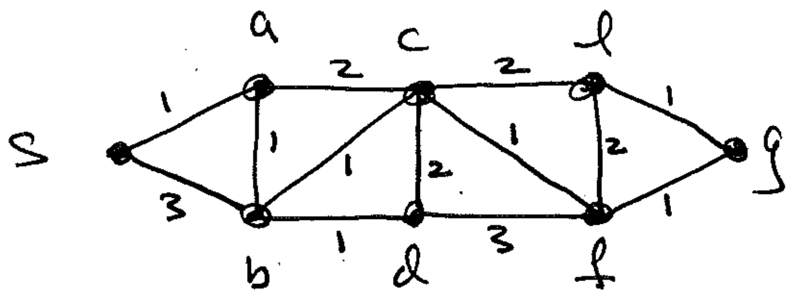
- T CONTAINS THE VERTICES WHOSE LABEL may change.
- $V - T$ CONSISTS OF COMPLETED VERTICES.



Ex.



	s	a	b	c	d	e	f	g	T
λ	0	∞	∞	∞	∞	∞	∞	∞	$\{s,a,b,c,d,e,f,g\}$
	0	1	3	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$
	0	1	2	3	∞	∞	∞	∞	$\{b,c,d,e,f,g\}$
	0	1	2	3	3	∞	∞	∞	$\{c,d,e,f,g\}$
	0	1	2	3	3	5	4	∞	$\{d,e,f,g\}$
	0	1	2	3	3	5	4	∞	$\{e,f,g\}$
	0	1	2	3	3	5	4	5	$\{e,g\}$
	0	1	2	3	3	5	4	5	$\{g\}$
	0	1	2	3	3	5	4	5	$\emptyset$

Thm

When Dijkstra is complete, $\lambda(v) = \delta(s,v)$ for all $v \in V$.

Rmks:

- λ-values never increase.
- After u is deleted from T, $\lambda(u)$ never changes.

Lemma

If after Dijkstra is complete, $\lambda(v) = \infty$, then $\delta(s,v) = \infty$, i.e. v is not reachable from s.

Proof is similar to corresponding fact re: BFS. earlier.

Lemma:

If at some point $\lambda(v)$ becomes finite, then G contains an s-v path of weight $\lambda(v)$.

Proof: use induction on time (i.e. the particular step) at which $\lambda(v)$ is assigned a finite value.

On first step, $\lambda(s) = 0$, and the trivial path from s to s has weight 0. This completes base case.

ASSUME $v \neq s$ AND THAT $\lambda(v)$ BECOMES FINITE AT SOME TIME DURING EXECUTION OF DIJKSTRA.

IND. HYP:

ASSUME THAT FOR EVERY FINITE LABEL $\lambda(u)$ ASSIGNED BEFORE $\lambda(v)$, THERE EXISTS AN s-u PATH IN G OF WEIGHT $\lambda(u)$.

AT TIME $\lambda(v)$ BECOMES FINITE (OR ANY OTHER TIME $\lambda(v)$ steps DOWN IN VALUE) WE SET IN (8):

$$\lambda(v) = \lambda(u) + w(uv)$$

where u is ADJACENT TO v, AND HAS BEEN ASSIGNED A (NECESSARILY FINITE) LABEL $\lambda(u)$ PRIOR TO THAT OF v.

By ind. Hyp. G CONTAINS AN s-u PATH OF weight $\lambda(u)$. APPEND TO THIS PATH THE EDGE uv, TO GET AN s-v PATH OF weight $\lambda(u) + w(uv) = \lambda(v)$. ///

<u>LEMMA</u>

AT THE TIME u IS CHOSEN ON line (6) OF DIJKSTRA, WE HAVE $\lambda(u) = \delta(s,u)$.

<u>RMK.</u>

THIS PROVES THE CORRECTNESS OF DIJKSTRA, SINCE WE CAN NEVER HAVE $\lambda(u) < \delta(s,u)$, BY LAST lemma.

EXERCISE: READ PROOF IN BOOK & IN COURSE NOTES.

<u>NOTE</u>: PROOF USES (IN AN ESSENTIAL WAY) THE FACT THAT $w(e) \geq 0$ for all $e \in E(G)$.

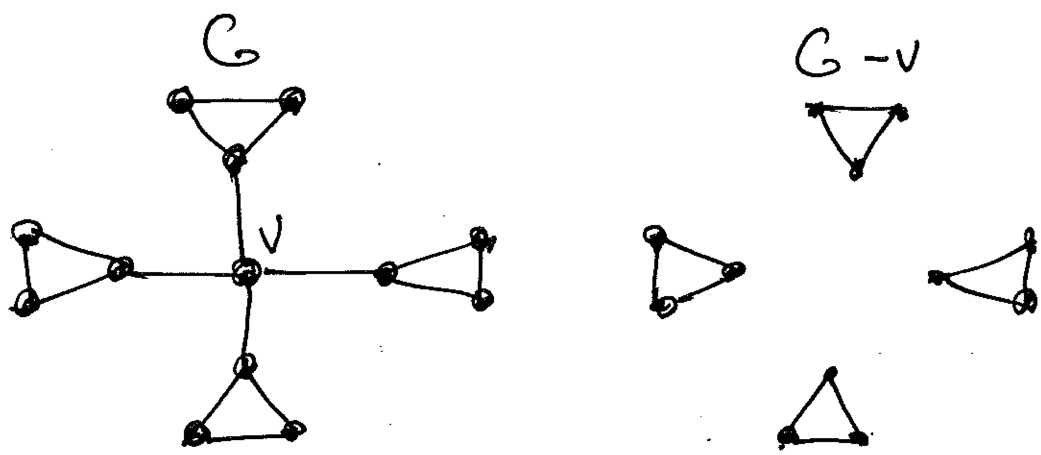
(2.6) CUT VERTICES

Defn

WE CALL $v \in V(G)$ A CUT VERTEX (ALSO ARTICULATION POINT) IF

$$\omega(G-v) > \omega(G)$$

Ex.



Thm

LET G BE CONNECTED. THEN $v \in V(G)$ IS A CUT VERTEX IFF THERE EXIST VERTICES u & w (OTHER THAN v) S.t. v LIES ON EVERY u-w PATH.

Rmk RESULT HOLDS FOR DISCONN. GRAPHS BY APPLYING THM TO $\mathrm{Comp}_G(v)$.

PROOF:

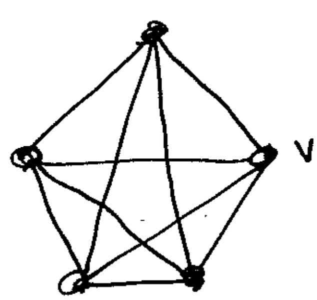
(⇒) SUPPOSE v IS A C.V. IN G, SO $G-v$ IS DISCONN. PICK u, w FROM DIFFERENT COMPONENTS OF $G-v$. ALTHOUGH G CONTAINS A u-w PATH (BEING CONNECTED), $G-v$ DOES NOT. THUS EVERY u-w PATH IN G MUST CONTAIN v.

(⇐) SUPPOSE $u, w \in V(G)-\{v\}$ ARE SUCH THAT EVERY u-w PATH IN G INCLUDES v. THEN $G-v$ CAN CONTAIN NO u-w PATH. ∴ u, w LIE IN DIFFERENT COMPONENTS OF $G-v$. ∴ $G-v$ IS DISCONNECTED. ///

RMKS

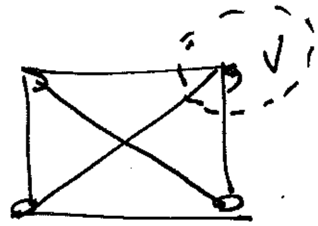
OBSERVE K_n HAS <u>NO</u> CUT VERTICES.

EX. K_5

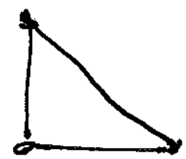


$K_5 - v \cong K_4$

K_4



$K_4 - v \cong K_3$



AT THE OTHER EXTREME HAVE P_n

P_n

$\#$ CUT VERTICES $= n-2$.

THM
LET G HAVE n VERTICES $(n \geq 2)$. THEN G HAS AT MOST $n-2$ CUT VERTICES.

PROOF

IF RESULT HOLDS FOR CONN. GRAPHS, IT HOLDS FOR DISCONN. GRAPHS. WE ASSUME NOW THAT G IS CONNECTED.

ASSUME, TO GET A $\times$, THAT G CONTAINS AT LEAST $n-1$ CUT VERTICES, i.e. G CONTAINS AT MOST 1 NON-CUT VERTEX.

PICK $u, v \in V(G)$ s.t.

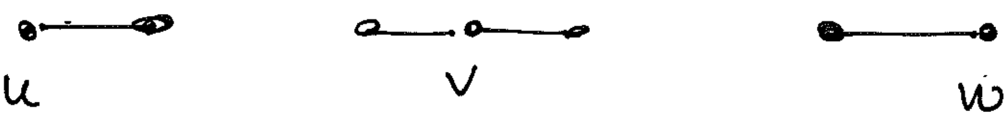
$$d(u,v) = \mathrm{Diam}(G) = \max\{d(x,y) \mid x, y \in V(G)\}.$$

SINCE G IS CONN. & $n \geq 2$ WE KNOW $u \neq v$. BY OUR ASSUMPTION, (at least) ONE OF u & v IS A CUT VERTEX IN G, SAY v IS A CUT VERTEX.

$\therefore$ $G-v$ IS DISCONN. PICK $w \in V(G)$ lying IN A COMPONENT OF $G-v$ OTHER THAN THAT OF u.

$\therefore$ EVERY u-w PATH INCLUDES v.

IT FOLLOWS THAT A SHORTEST u-w PATH (PROPERLY) CONTAINS A SHORTEST u-v PATH



$\therefore d(u,v) < d(u,w)$

THIS CONTRADICTS OUR CHOICE OF u & v, i.e. $d(u,v) = \text{Diam}(G)$.

$\therefore$ OUR ASSUMPTION WAS FALSE

$\therefore$ # CUT VERTICES $\leq n-2$. ///.

<u>Defn</u>

LET G BE CONN. A SET $W \subseteq V(G)$ IS CALLED A <u>SEPARATING SET</u> iff $G-W$ IS DISCONNECTED.

<u>NOTE</u>: K_n HAS NO SEPARATING SET.

Defn

Let G BE SIMPLE & CONN.
THE (VERTEX) CONNECTIVITY OF G
~~ISO~~ IS THE SIZE OF A SMALLEST
SEPARATING SET

NOTATION $\kappa(G)$.

SO $\kappa(G) = \min\{|W| : W \subseteq V(G)$ IS SEP.$\}$

Define: $\kappa(K_n) = n-1$, i.e. THE
OF VERTICES WHOSE REMOVAL
GIVES K_1.