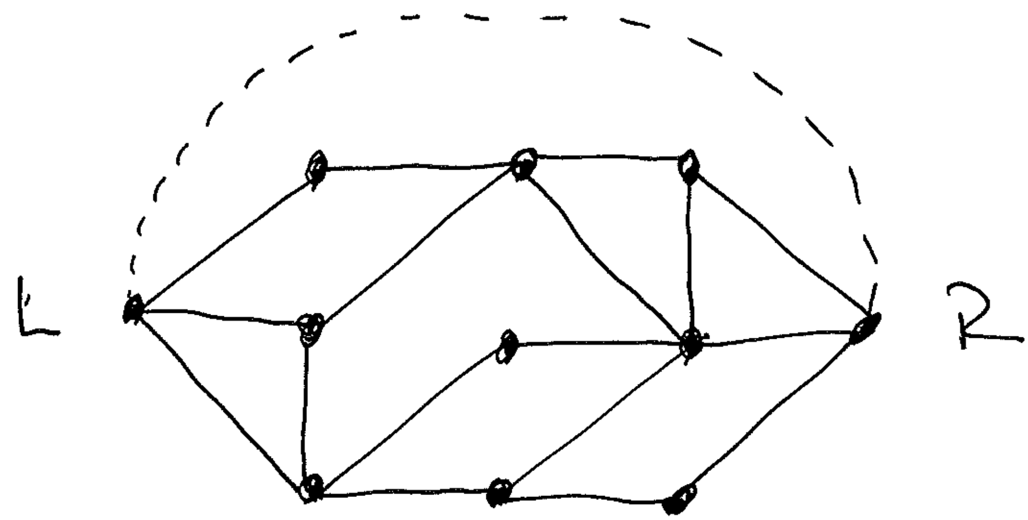
CMPE 177 7-20-09



TO FIND A NEAR TREE: INSERT A NEW EDGE FROM L TO R, DETERMINE A SPANNING TREE THAT INCLUDES THAT NEW EDGE, THEN REMOVE THE NEW EDGE. THE RESULT IS A NEAR TREE, i.e. A SPANNING FOREST WITH TWO COMPONENTS, ONE OF WHICH CONTAINS L, THE OTHER R.

(2.5) SHORTEST PATH PROBLEMS

SINGLE SOURCE SHORTEST PATH (SSSP)

PROBLEM: GIVEN $s \in V(G)$ (<u>SOURCE</u>)

FIND A SHORTEST s-v PATH FOR ALL $v \in V(G)$ (IF SUCH A PATH EXISTS)

<u>BREADTH FIRST SEARCH (BFS)</u>:

- ASSIGNS A LABEL $\lambda(v)$ TO EVERY $v \in V(G)$ SUCH THAT $\lambda(v) = d(s, v)$.
- AFTER LABELS, CAN BACKTRACK TO FIND SHORTEST PATHS.

BFS (G, s)

1.) $\lambda(s) \leftarrow i \leftarrow 0$

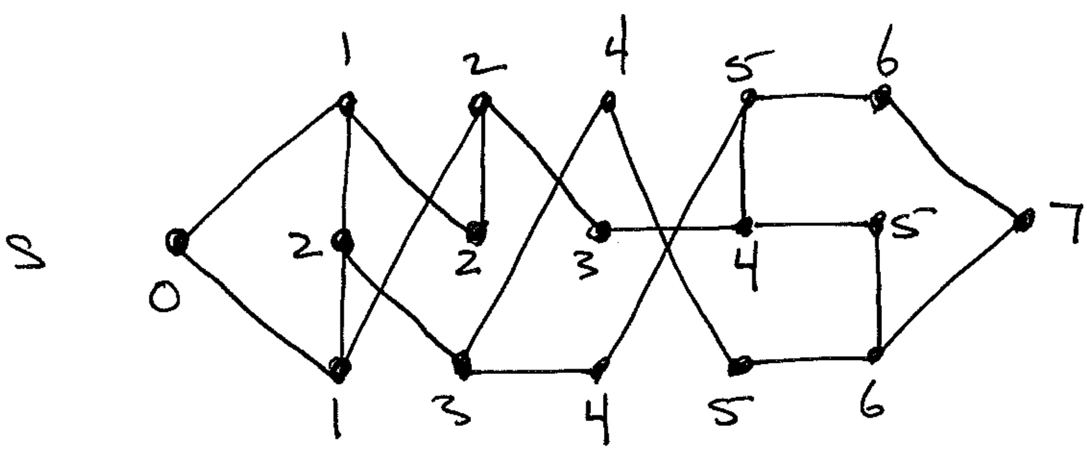
2.) while there are any vertices labeled i with unlabeled neighbors

3.) label all such neighbors $i+1$

4.) $i \leftarrow i+1$

5.) label all remaining vertices ∞.

Ex.



THM: When BFS is complete,
$\lambda(v) = d(s,v)$ for all $v \in V(G)$.

PROOF:

If $\lambda(v) = \infty$, THEN ALL NEIGHBORS OF v MUST BE LABLED ∞ (OTHERWISE v WOULD HAVE BEEN ASSIGNED A FINITE label ON SOME INSTANCE OF (3).) WE CAN SEE THAT ANY VERTEX REACHABLE FROM v must Also BE ∞. THUS ALL VERTICES IN $C(v)$ ARE ∞. BUT $\lambda(s) = 0$, SO $s \notin C(v)$ $\therefore$ $d(s,v) = d(v,s) = \infty = \lambda(v)$.

NOW SUPPOSE $\lambda(v) < \infty$. WE USE INDUCTION ON $i = \lambda(v)$, i.e. WE show for All $i = 0, 1, 2, \ldots\ldots$ THAT

$$\lambda(v) = i \quad \text{IFF} \quad d(s,v) = i$$

BASE CASE: $i = 0$.

THERE IS ONLY ONE $v \in V(G)$ WITH $\lambda(v) = 0$, NAMELY $v = s$. INDEED $d(s,s) = 0$.

LET $i > 0$ AND ASSUME THAT $\forall v$:

$\forall j < i$: $\lambda(v) = j$ IFF $d(s,v) = j$

MUST SHOW THAT $\forall v$:

$$\lambda(v) = i \ \overset{\longrightarrow}{\underset{\longleftarrow}{\text{IFF}}}\ d(s,v) = i$$

($\Rightarrow$) let $\lambda(v) = i$. THEN v WAS LABELED ON ITERATION $i-1$ OF LOOP 2-4, when IT WAS THE UNLABELED NEIGHBOR OF SOME $u \in V(G)$ WITH $\lambda(u) = i-1$. By IND. HYP. $d(s,u) = i-1$.

NOW APPEND EDGE uv TO A SHORTEST s-u PATH TO OBTAIN AN s-v PATH OF length i

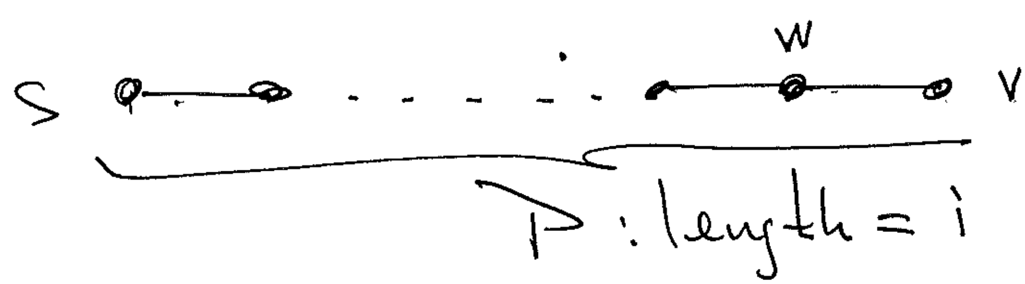
THUS $d(s,v) \le i$.

BUT ALSO $d(s,v) \ge i$ SINCE IF $d(s,v) = j < i$, THEN BY INDUCTION HYP $\lambda(v) = j < i$, BUT $\lambda(v) = i$ ✗.

$\therefore$ $d(s,v) = i$.

($\Leftarrow$) SUPPOSE $d(s,v) = i$. THEN THERE EXISTS A SHORTEST s-v PATH, CALL IT P, WITH $\text{length}(P) = i$.

let w BE NEXT TO LAST VERTEX ALONG P:



WE MUST HAVE $d(s,w)=i-1$ (SINCE ANY SUB-PATH OF A SHORTEST PATH IS A SHORTEST PATH.) BY THE IND. HYP. $\lambda(w)=i-1$. NOW v MUST ITSELF BE UNLABELED BEFORE ITERATION $i-1$ OF LOOP 2-4. (SINCE OTHERWISE $d(s,v)<i$, BY IND HYP.) SINCE v IS ADJACENT TO w, v IS ASSIGNED $\lambda(v)=i$ ON ITERATION $i-1$ OF LOOP 2-4.

$\therefore \lambda(v)=i$. ///

HOW TO FIND SHORTEST PATHS?

Let $v \in V(G)$, $k = \lambda(v)$ AFTER BFS.

TO FIND AN s-v PATH OF length k

DO:

1.) $v_k \leftarrow v$, $i \leftarrow k$
2.) while $i > 0$
3.) choose u ADJ TO v_i with $\lambda(u) = i-1$.
4.) $v_{i-1} \leftarrow u$
5.) $i \leftarrow i-1$

THEN $P: s = v_0, v_1, v_2, \ldots, v_k = v$ is A SHORTEST s-v PATH.

TO FIND THE # OF SHORTEST s-v PATHS, COMPUTE sv^{TH} ENTRY IN A^k, WHERE A is THE ADJ. MATRIX.

Recall $A^k = (a_{ij}^{(k)})$ and

$$a_{sv}^{(k)} = \sum_{\{i_1, \ldots, i_{k-1}\}} a_{si_1} a_{i_1 i_2} \cdots a_{i_{k-1} v}$$

$$= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} a_{si_1} a_{i_1 i_2} \cdots a_{i_{k-1} v}$$

We compute this formula in stages by assigning a new label $\mu(u)$ for all $u \in V(G)$ with $\lambda(u) \leq k$.

Inputs (s, v) :

1.) $\mu(v) \leftarrow 1$

2.) for all $u \neq v$ with $\lambda(u) = k$

3.) $\quad \mu(u) \leftarrow 0$

4.) $i \leftarrow k$

5.) while $i > 0$

6.) $\quad$ for u with $\lambda(u) = i-1$

7.) $\quad\quad \mu(u) \leftarrow \sum_{\substack{w \\ \lambda(w)=i}} a_{uw} \cdot \mu(w)$

8.) $\quad i \leftarrow i-1$

EX. labels : (λ, μ) .