CMPE 177 7-13-09

- MIDTERM 1 : Wed 7-15-09
  5:00 - 6:10
  + 10 min break

- TA SESSION : TUE 7-14-09
  5:00 - 7:00

MATRIX TREE THEOREM

For Any r : $1 \leq r \leq n$

$$|S(G)| = \det(L_{rr})$$

$(n-1)\times(n-1)$

where

$$L = \underset{n \times m}{\vec{M}} \, \underset{m \times n}{\vec{M}^T} = D - A$$

RECALL: <u>CAUCHY-BINET</u> THM:

Let $s \leq t$, P an $s \times t$ matrix, Q a $t \times s$ matrix. Let

$$I = \{ i_1 < i_2 < \dots < i_s \}$$

P_I = $s \times s$ submatrix of P with cols indexed by I.

Q^I = $s \times s$ submatrix of Q with rows indexed by I.

THEN

$$\det(PQ) = \sum_I \det(P_I) \cdot \det(Q^I)$$

<u>EX</u> $s = 2$, $t = 3$

$$P = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} -1 & 1 \\ 1 & 2 \\ 3 & 0 \end{pmatrix}$$

$$PQ = \begin{pmatrix} 5 & 1 \\ 2 & -2 \end{pmatrix} \quad \det(PQ) = \boxed{-12}$$

$$I = \{1, 2\}$$

$$\left.\begin{aligned} P_I &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \det P_I = -1 \\ Q^I &= \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \quad \det Q^I = -3 \end{aligned}\right\} \times \qquad 3$$

$$I = \{1, 3\}$$

$$\left.\begin{aligned} P_I &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \det P_I = 1 \\ Q^I &= \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} \quad \det Q^I = -3 \end{aligned}\right\} \times \qquad -3$$

$$I = \{2, 3\}$$

$$\left.\begin{aligned} P_I &= \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} \quad \det P_I = 2 \\ Q^I &= \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \quad \det Q^I = -6 \end{aligned}\right\} \times \qquad -12$$

$$\sum_I \det P_I \cdot \det Q^I = 3 - 3 - 12 = \boxed{-12}$$

PROOF OF MTT:

LET $\vec{M}_r$ BE THE $(n-1)\times m$ MATRIX OBTAINED BY DELETING r^{th} row from $\vec{M}$. THEN

$$L_{rr} = \underbrace{\vec{M}_r}_{(n-1)\times m} \underbrace{\vec{M}_r^T}_{m\times(n-1)}$$

LET $s = n-1$ & $t = m$ IN Cauchy-Binet. (NOTE: $n-1 \leq m$ SINCE G is CONN. $\therefore$ $s \leq t$.) so By C.B.

$$\det(L_{rr}) = \sum_N \det N \cdot \det N^T = \sum_N (\det N)^2$$

where N runs over all $(n-1)\times(n-1)$ SUBMATRICES OF $\underbrace{\vec{M}_r}_{(n-1)\times m}$.

EACH SUCH MATRIX N CORRESPONDS TO A CHOICE OF $(n-1)$ Columns in $\vec{M}_r$, hence also in $\vec{M}$, and hence to a set of $(n-1)$ EDGES in G.

WE will show THAT $\det N = \pm 1$ iff those EDGES form a spanning TREE in G, AND OTHERWISE $\det N = 0$. Result follows!

FIRST SUPPOSE THE $n-1$ EDGES CORRESPONDING TO N <u>DO NOT</u> form A TREE. $\therefore$ THE SPANNING SUBGRAPH with THOSE EDGE MUST BE DISCONNECTED.

THIS SUBGRAPH HAS A CONN. COMPONENT which DOES NOT CONTAIN v_r

THE ROWS OF N CORRESPONDING TO THE VERTICES IN THIS COMPONENT CONSTITUTE THE ORIENTED INCIDENCE MATRIX for that component.

So these rows, as vectors, add to the zero vector. $\therefore N$ is SINGULAR i.e. $\det N = 0$.

NOW SUPPOSE THE SPANNING SUBGRAPH CORRESPONDING TO N forms a tree.

Choose indices $i_1, i_2, \ldots, i_{n-1}$ & $j_1, j_2, \ldots, j_{n-1}$ as follows:

Pick a leaf $v_{i_1} \neq v_r$ in this tree, with e_{j_1} as its (sole) incident edge. Delete v_{i_1} & e_{j_1} TO GET A NEW TREE.

AGAIN THERE is a leaf $v_{i_2} \neq v_r$ AND INCIDENT EDGE e_{j_2}, which we Delete.

Continue in this way until VERTICES $v_{i_1}, \ldots, v_{i_{n-1}}$ AND EDGES $e_{j_1}, \ldots, e_{j_{n-1}}$ HAVE BEEN DETERMINED.

NOW RE-ARRANGE ROWS & COLS OF N SO THAT v_{i_k} CORRESPONDS TO ROW k & e_{j_k} TO COL k $(1 \leq k \leq n-1)$.

OBSERVE THAT v_{i_k} is NOT INCIDENT INCIDENT WITH e_{j_ℓ} for $1 \leq k < \ell \leq n-1$.

HENCE N is lower TRIANGULAR.

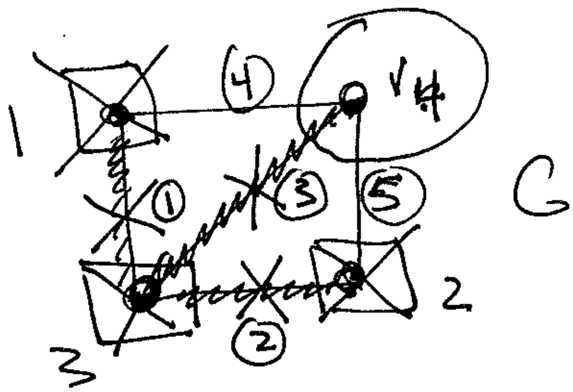
FURTHER v_{i_k} IS INCIDENT WITH l_{j_k},

SO THE kk^{TH} entry IN N

$(1 \le k \le n-1)$ IS $+1$ OR -1.

$$\therefore \det N = \pm 1$$

Illustration:



$$\vec{M} = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \leftarrow 4 = n$$

$$\vec{M}_4 = \begin{array}{c} \begin{array}{ccccc} (1) & (2) & (3) & & \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 \end{pmatrix} \end{array}$$

$$N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \quad \det N = (-1)(1)(-1) = -1$$

$\therefore$ $N \sim$ tree iff $\det N = \pm 1$

$N \sim$ non-tree iff $\det N = 0$

$$\therefore \det(L_{rr}) = \sum_{N} (\det N)^2 = \#\ \text{sp. trees.}$$

///.

<u>Cayley's Theorem</u>:

$$|S(K_n)| = n^{n-2}$$

PROOF: THE LAPLACIAN OF K_n IS

$$L = D - A = \begin{pmatrix} (n-1) & -1 & \cdots & -1 \\ -1 & (n-1) & \cdots & -1 \\ \vdots & \vdots & & \\ -1 & -1 & \cdots & (n-1) \end{pmatrix} \quad n \times n$$

AND

$$L_{rr} = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix} \quad (n-1) \times (n-1)$$

Recall: Can add a ROW (resp. col) TO ANOTHER ROW (RESP. COL) WITHOUT CHANGING DETERMINANT.

- SUBTRACT COL. 1 from all others

$$\begin{pmatrix} n-1 & -n & -n & \cdots & -n \\ -1 & n & 0 & \cdots & 0 \\ -1 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & n \end{pmatrix} \quad (n-1)\times(n-1)$$

- ADD EVERY ROW TO ROW 1

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & n & 0 & \cdots & 0 \\ -1 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & n \end{pmatrix} \quad (n-1)\times(n-1)$$

This last matrix has $\det = n^{n-2}$

$$\therefore \ |S(K_n)| = \det(L_{rr}) = n^{n-2} \quad ///.$$

(2.4) Minimum Weight Sp. Trees

Defn

A *weighted graph* is a graph $G = (V, E)$ with a real function

$$w : E \to \mathbb{R}$$

The *weight* of a subgraph $H \subseteq G$ is:

$$w(H) = \sum_{e \in E(H)} w(e)$$

Defn A *minimum weight spanning tree* (MWST) T satisfies

$$w(T) \leq w(S)$$

for all sp. trees S in G.

RECALL 'BOTTOM UP' Algorithm for CONSTRUCTING A SP. TREE.

LET $E' = E - \{loops\}$.

1.) $F \leftarrow \phi$

2.) while $|F| < n-1$

3.) choose $e \in E' - F$ such that $(V, F \cup \{e\})$ is acyclic.

4.) $F \leftarrow F \cup \{e\}$

OBSERVE (3) CAN BE PERFORMED AS LONG AS $|F| < n-1$, SINCE we can choose e TO BE any edge joining 2 COMPONENTS (V, F). THEN e is A BRIDGE IN $(V, F \cup \{e\})$, AND SO DOES NOT CREATE A Cycle.

If no such e exists then (V, F) is conn & acyclic by its very construction. ∴ $|F| = n-1$ by treeness thm, contradicting the while loop condition. ∴ such an e must exist.

<u>THM</u>

When this algorithm is complete $T=(V,F)$ is a sp. tree in G.

<u>Proof:</u>

Algorithm stops when $|F| = n-1$. $T=(V,F)$ is acyclic by construction. ∴ T is a tree by the treeness thm.

OBSERVE SP. TREES HAVE A DUAL CHARACTERIZATION

(i) minimal W.R.T. CONNECTEDNESS

(ii) maximal W.R.T. Acyclic PROPERTY.

A slight ALTERATION OF LAST ALGORITHM will GIVE A MWST.

<u>KRUSKAL</u>

1.) $F \leftarrow \emptyset$

2.) while $|F| < n-1$

3.) choose $e \in E' - F$ SUCH THAT

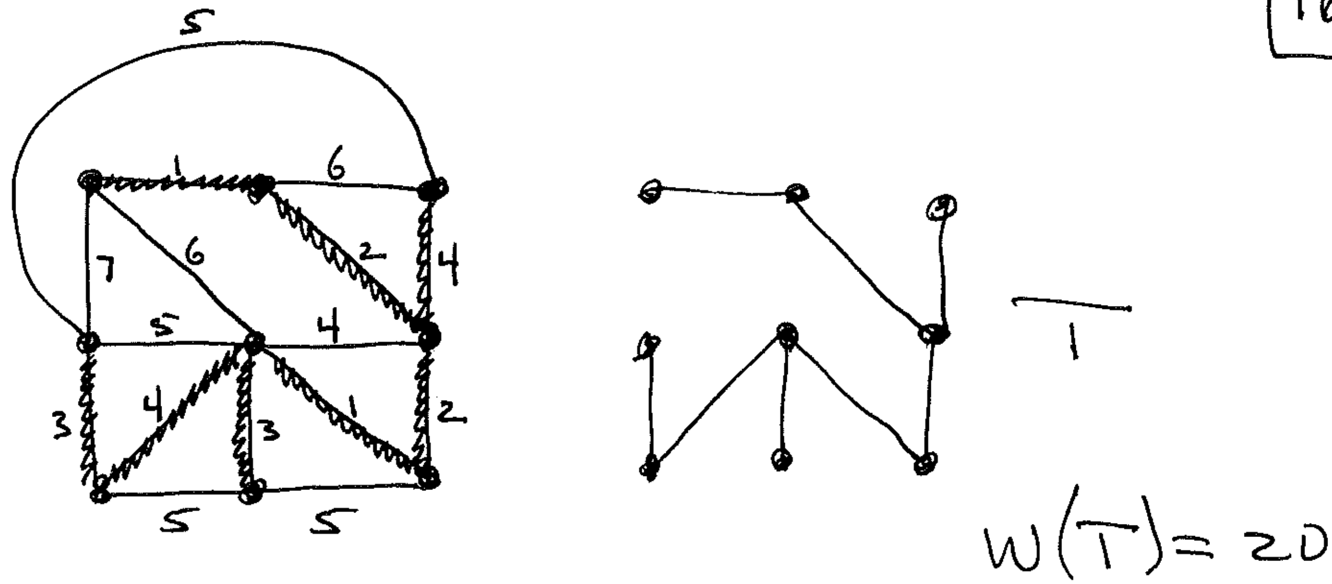
(i) $(V, F \cup \{e\})$ is Acyclic

(ii) $w(e)$ is minimum SUBJECT TO (i)

4.) $F \leftarrow F \cup \{e\}$

(SEE C.L.R.S. version of Kruskal)

Ex



Thm

when Kruskal is complete, $T=(V,F)$ is a m.w.s.t. in G.