CMPE 177 7-1-09

INCIDENCE MATRIX : $M = M(G)$

is an $n \times t$ matrix, where $n = |V(G)|$ and $t = |E(G)|$, say

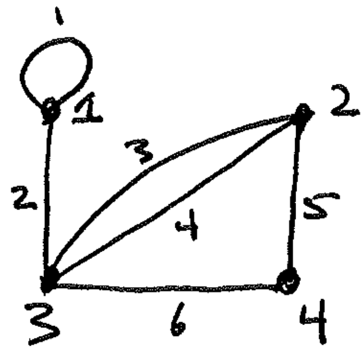
$$V(G) = \{v_1, \dots, v_n\}$$
$$E(G) = \{e_1, \dots, e_t\}$$

$$M = (m_{ij}) \qquad 1 \le i \le n, \ 1 \le j \le t$$

where

$$m_{ij} = \begin{cases} 0 & v_i \text{ not inc. with } e_j \\ 1 & v_i \text{ is inc. w. } e_j, \text{ not loop} \\ 2 & v_i \text{ is inc. w. } e_j, \text{ is loop.} \end{cases}$$

Ex.



$$M = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

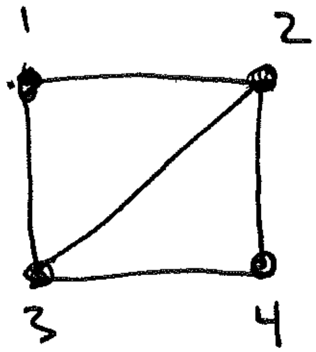
NOTE:

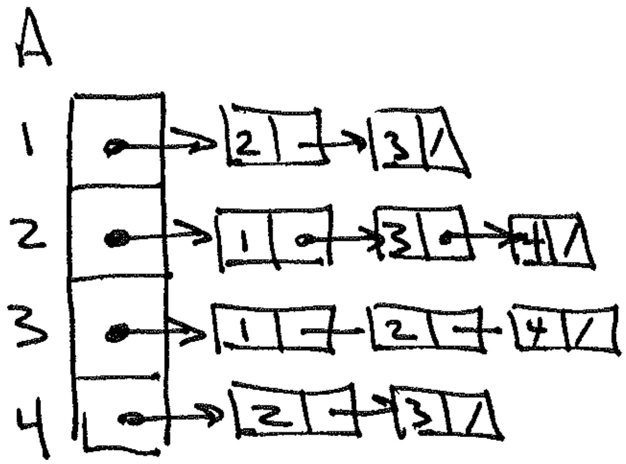
- Sum of a row is deg of vertex

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

ADJACENCY LIST REPRESENTATION

Ex

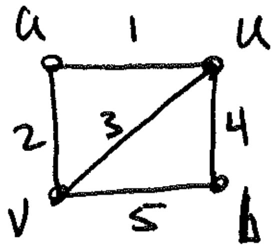




(1.8) Fusion

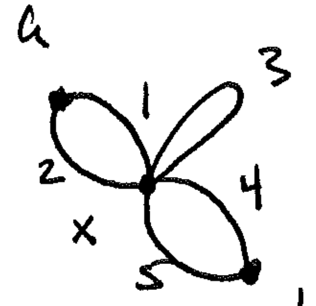
Let $u, v \in V(G)$, $u \neq v$. $uv \in E(G)$. We can *fuse* or *identify* u with v, to obtain a new graph G_1: replace u, v with a single new vertex x which is adjacent to all neighbors of u, and all neighbors of v. Edge uv becomes a loop in G_1.

Ex.



G

$\xrightarrow{\text{Fuse}(uv)}$



G_1

$$\begin{array}{c|cccc} & a & u & v & b \\ \hline a & & & & \\ u & & & & \\ v & & & & \\ b & & & & \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} a & u & v & b \end{array} \\ \begin{array}{c} a \\ u \\ v \\ b \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$A(G)$

$$\begin{array}{cc} & \begin{array}{ccc} a & x & b \end{array} \\ \begin{array}{c} a \\ x \\ b \end{array} & \begin{pmatrix} 0 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \end{array}$$

$A(G_1)$

TWO STEPS:

(1) Replace u's row (resp. column) by sum of u's row (column) and v's row (resp. column);

(2) Delete v's row & column.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{(1)} \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

FINDING CONN. COMPONENTS

- FUSION
- REPLACE A GRAPH BY IT'S UNDERLYING SIMPLE GRAPH. (USG)

ALGORITHM

1.) REPLACE G BY ITS U.S.G.

2.) while there is a non-isolated $v \in V(G)$

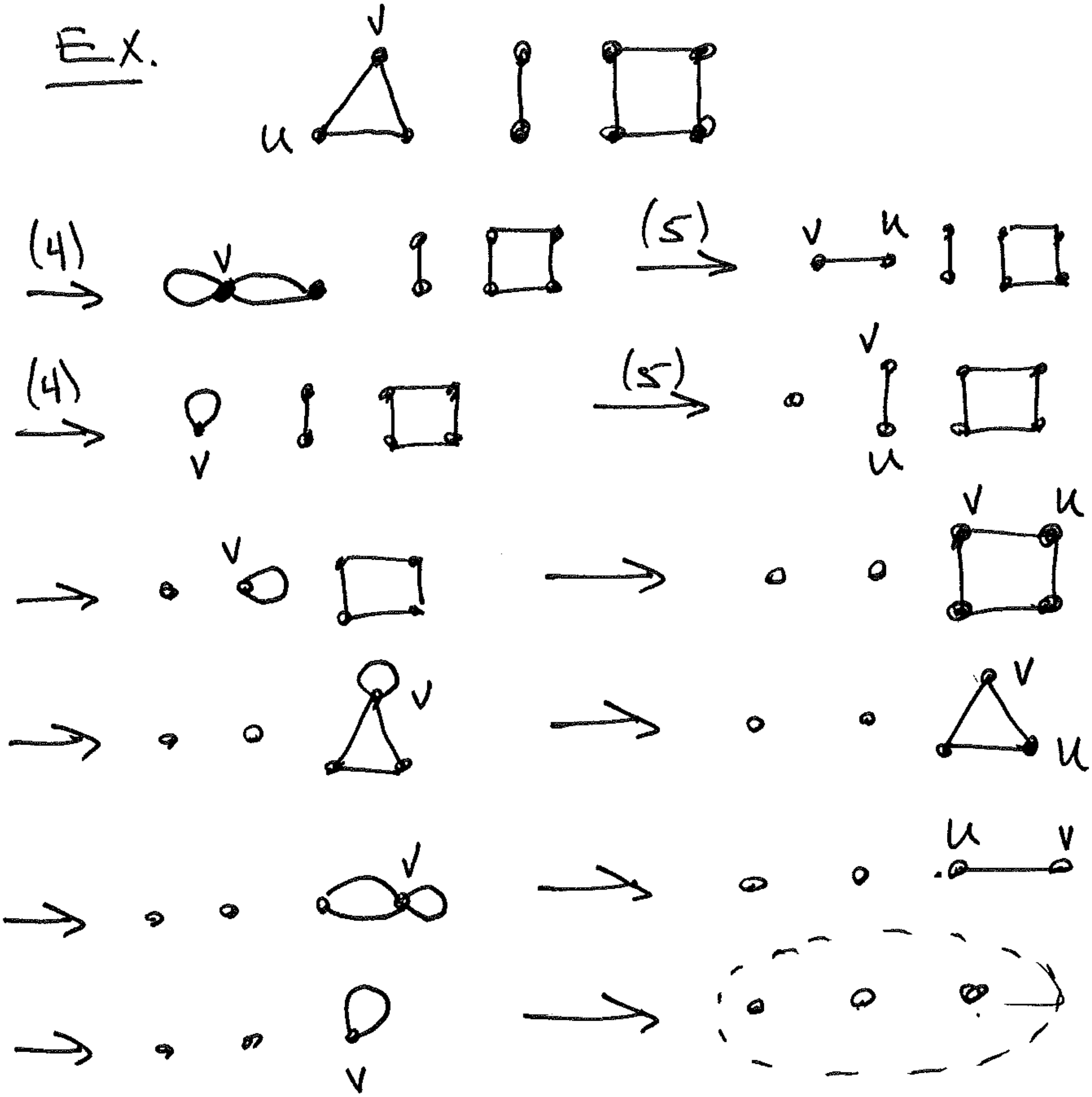
3.) while there is $u \in V(G)$ adj to v

4.) FUSE u TO v, CALL new vertex v

5.) REPLACE G BY ITS USG.

When COMPLETE G is A NULL GRAPH (i.e. NO EDGES). THE NUMBER OF VERTICES left is THE # OF CONN. COMP. IN ORIGINAL GRAPH.

Ex.



$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

EXERCISE: WRITE DOWN CORRESPONDING ADJ MATRIX OPS.

(2.1) TREES

Defn A GRAPH is CALLED *ACYCLIC* IFF IT CONTAINS NO CYCLES (i.e. NO SUBGRAPHS HOMOMORPHIC TO C_K.) ALSO CALLED A *FOREST*.

Defn A *TREE* IS A GRAPH WHICH IS BOTH CONNECTED & ACYCLIC.

NOTE: ACYCLIC $\Longrightarrow$ SIMPLE
implies

C_1  C_2 

EX.

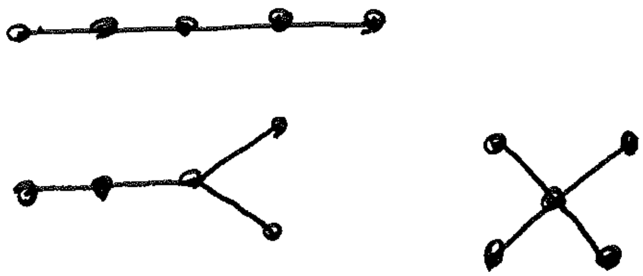
$n=1$: 

$n=2$: 

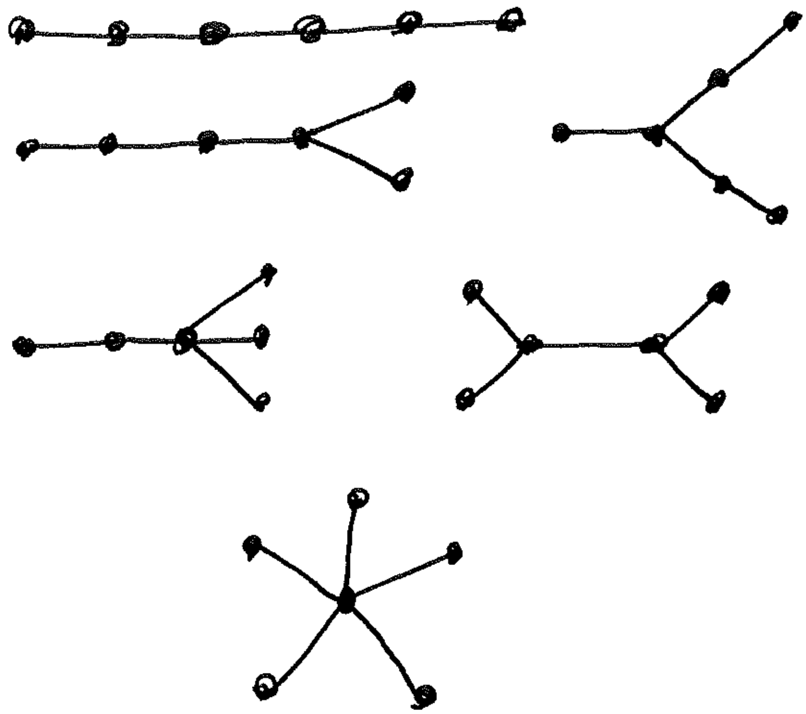
$n=3$: 

$n=4$:

$n=5$:



$n=6$:



$n=7$: EXERCISE

Lemma

IF VERTICES OF G ALL HAVE DEGREE AT LEAST 2, THEN G CONTAINS A CYCLE.

Corollary: ANY TREE (WITH $n \geq 2$) MUST CONTAIN SOME VERTICES OF Degree 1, CALLED LEAVES.

Proof of lemma:

WE ASSUME G IS SIMPLE, SINCE OTHERWISE CONCLUSION IS TRIVIALLY TRUE.

LET $x_0, x_1 \in V$ BE ADJ. IN G.
CHOOSE $x_2 \in V$ TO BE ANY VERTEX ADJACENT TO x_1, OTHER THAN x_0.
(x_2 EXISTS BY HYPOTHESIS, i.e. $d(x_1) \geq 2$ AND G IS SIMPLE, SO x_2 HAS AT LEAST 2 NEIGHBORS.)

PICK $x_3, x_4, x_5, \ldots,$ IN A SIMILAR MANNER SO THAT x_{i+1} IS ADJ. TO x_i, AND $x_{i+1} \neq x_{i-1}$.

CONTINUE UNTIL SOME VERTEX x_k IS REPEATED, i.e. $x_k = x_{k+m}$ (MUST OCCUR SINCE V IS FINITE)

THEN THE SUB-SEQ

$$x_k, x_{k+1}, \ldots\ldots, x_{k+m-1}, x_{k+m} = x_k$$

IS THE REQUIRED cycle. ///.

(2.2) BRIDGES

Lemma

Let $e \in E(G)$. Then

$$\omega(G) \le \omega(G-e) \le \omega(G)+1$$

$\uparrow$

# of conn. comp. in G

Defn if $\omega(G-e) = \omega(G)+1$, the edge is called a **bridge**.

Proof:

If e is a loop, then obviously $\omega(G-e) = \omega(G)$, so we assume now that G has no loops. Let $e = uv$.

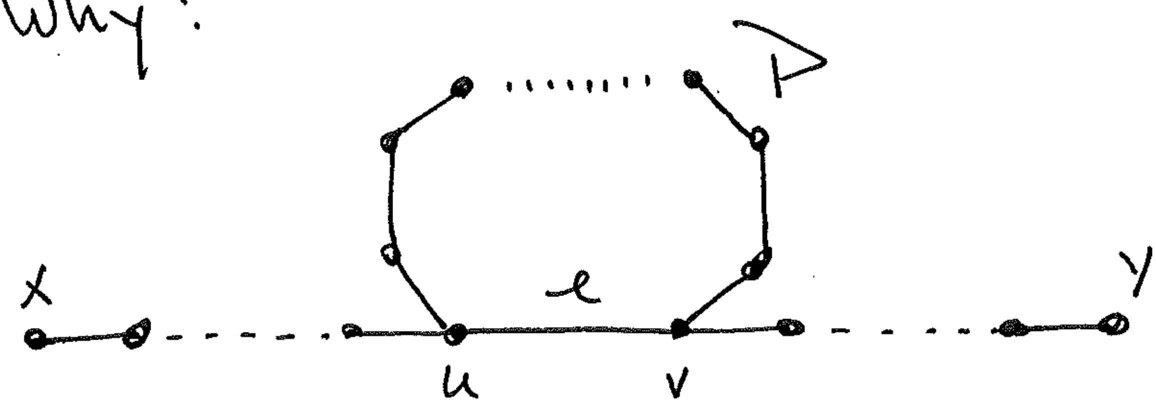
WE HAVE 2 CASES:

CASE 1:

THERE EXISTS A u-v PATH P IN G WHICH DOES NOT INCLUDE e.

IN THIS CASE $\omega(G-e) = \omega(G)$.

WHY?



SINCE ANY PATH IN $Comp_G(u) = G[C(u)]$ WHICH INCLUDES e CAN BE REPLACED BY ONE WHICH DOES NOT INCLUDE e BY SPLICING OUT e AND SPLICING IN P (AND DELETING SOME REDUNDANT EDGES IF NECESSARY.)

SO $C_G(u) = C_{G-e}(u)$, AND HENCE $comp_G(u) = comp_{G-e}(u)$. ALSO REMOVING e DOES NOT AFFECT OTHER COMPS, SO $\omega(G) = \omega(G-e)$.

<u>CASE 2</u>

ALL u-v PATHS IN G INCLUDE EDGE e.

~~(SINCE G IS SIMPLE THIS IS JUST ONE PATH $u \overset{e}{—} v$.)~~

i.e. THERE ARE NO EDGES PARALLEL TO $e = uv$

DON'T EXIST.

MUST SHOW IN THIS CASE

$$\omega(G-e) = \omega(G) + 1.$$

FIRST OBSERVE $C_{G-e}(u) \cap C_{G-e}(v) = \emptyset$

SINCE ANY u-v PATH IN G INCLUDES e.

ALSO $C_{G-e}(u) \cup C_{G-e}(v) = C_G(u)$

TO SEE THIS let $x \in C_G(u)$ BUT $x \notin C_{G-e}(u)$. THE x-u PATH IN G MUST HAVE e AS ITS LAST EDGE. DELETING e FROM THIS THIS PATH IS AN x-v PATH IN G-e

AN x-v PATH IN G-e

$\therefore\; x \in C_{G-e}(v)$

SO $\text{COMP}_G(u)$ BECOMES, UPON REMOVAL OF e, TWO COMPONENTS: $\text{COMP}_{G-e}(u)$ AND $\text{COMP}_{G-e}(v)$.

SINCE OTHER COMPONENTS ARE UNAFFECTED, $\omega(G-e) = \omega(G)+1$.

///.

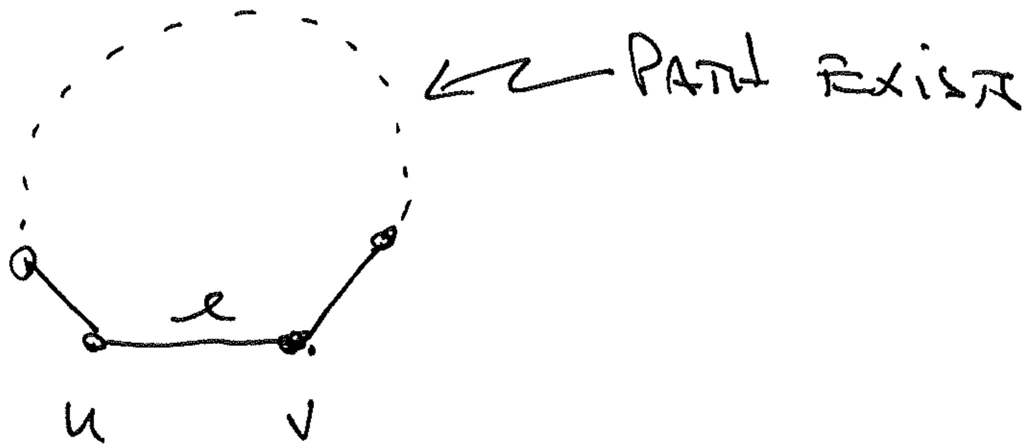
<u>Lemma</u>

IF G IS ACYCLIC, THEN EVERY EDGE IS A BRIDGE.

<u>PROOF:</u>

ASSUME (TO GET A CONTRADICTION) THAT $e \in E(G)$ IS A NON-BRIDGE. Let $e = uv$, i.e. u, v ARE THE NECESSARILY DISTINCT ENDS OF e. SINCE $G-e$ IS CONN. IT CONTAINS

A u-v PATH. ADDING e TO



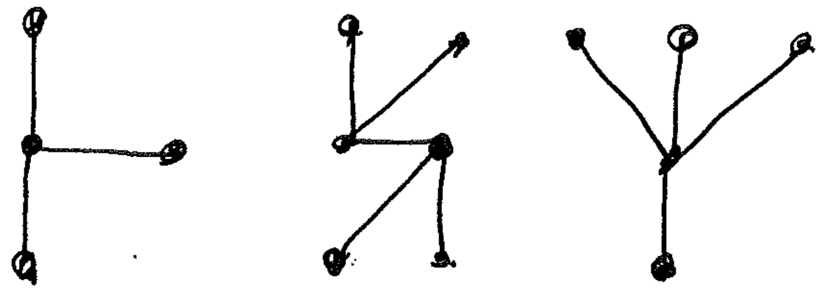
THIS PATH FORMS A CYCLE IN G, CONTRADICTING THAT G IS ACYCLIC.

THIS ※ (CONTRADICTION) SHOWS OUR ASSUMPTION WAS FALSE, SO NO SUCH EDGE EXISTS, SO EVERY EDGE IS A BRIDGE. ///

NOTE: CONVERSE IS ALSO TRUE. PROOF LATER.

EXAMPLES

Acyclic



NOTE: IN ANY GRAPH G, EVERY EDGE $e \in E$ IS ONE OF TWO TYPES

- e IS A BRIDGE.
- e BELONGS TO SOME CYCLE IN G.

PLEASE READ HANDOUT ON INDUCTION PROOFS.