CMPE 177 6-29-09

<u>Defn</u> WE SAY v IS REACHABLE FROM u IFF G CONTAINS A u-v PATH.

<u>Defn</u> G IS CALLED <u>CONNECTED</u> IFF for all $u, v \in V(G)$ v is Reachable from u. OTHERWISE G IS CALLED <u>DISCONNECTED</u>.

<u>Defn</u> Given $u \in V(G)$

$$C(u) = \{ v \in V(G) \mid v \text{ is REACHABLE from } u \}$$

<u>Defn</u> THE <u>CONNECTED COMPONENT</u> OF G CONTAINING u IS THE SUBGRAPH

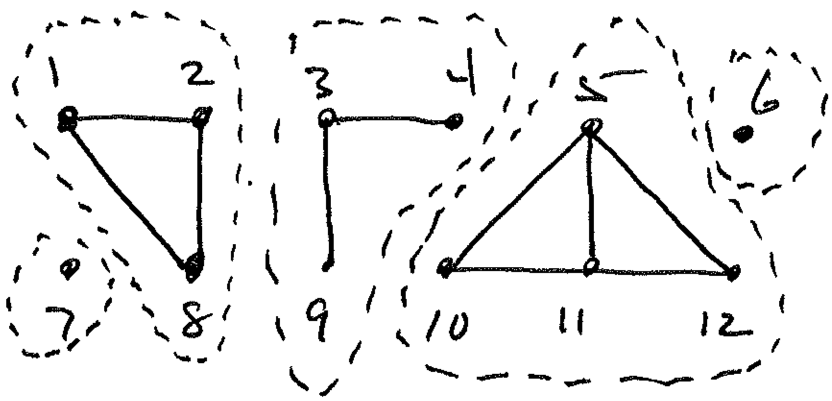
$G[C(u)]$, THE SUBGRAPH INDUCED BY $C(u)$.

EQUIVALENTLY: A CONNECTED COMPONENT IS A SUBGRAPH OF G WHICH IS

(1) CONNECTED

(2) MAXIMAL W.R.T. (1)

EX.



$\omega(G) = 5$

CONN BUT NOT MAXIMAL: $(\{1,2,8\}, \{12,18\})$

CONN & MAXIMAL: $(\{1,2,8\}, \{12,18,28\})$

NOTATION: $\omega(G) = \#$ CONN. COMPS.

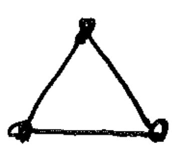
OBVIOUSLY G IS CONN. IFF $\omega(G) = 1$.

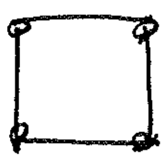
Defn The length of a path is the # of edges (likewise for a cycle.

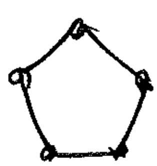
Defn A k-cycle is a cycle of length k. Up to isomorphism there is just one k-cycle, denoted C_k.

C_1 

C_2 

C_3 

C_4 

C_5 

P_k DENOTES A PATH OF LENGTH K.

P_0 

P_1 

P_2 

P_3 

$\vdots$

<u>THM</u> LET G BE A GRAPH WITH $|V(G)| \geq 2$. THEN G IS BIPARTITE IFF G CONTAINS NO ODD LENGTH CYCLE.

<u>PROOF</u>

($\Rightarrow$) SUPPOSE G IS BIPARTITE WITH BIPARTITION $V(G) = X \cup Y$.

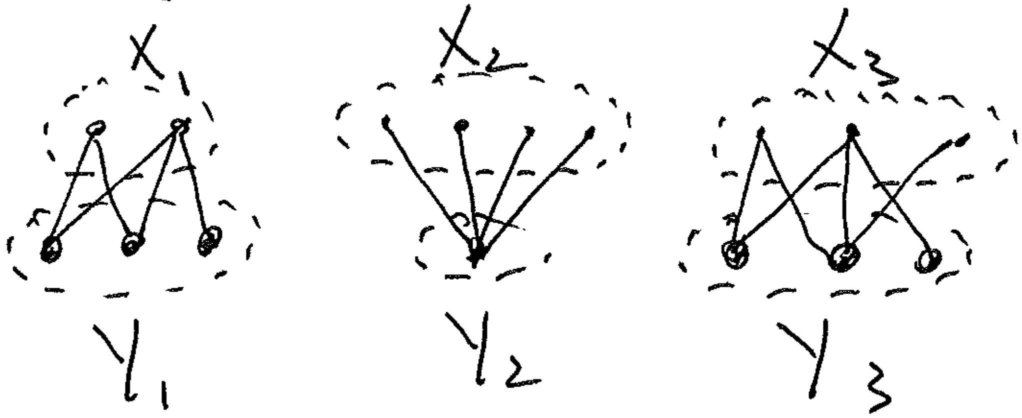
LET $C = v_0 v_1 v_2 \cdots \overset{v_0}{\overset{\parallel}{v_k}}$ BE A CYCLE IN G. MUST SHOW THAT ITS LENGTH, k, IS EVEN.

SUPPOSE $v_0 \in X$ (THE CASE $v_0 \in Y$ IS SIMILAR, AND WE OMIT IT.) THEN $v_1 \in Y$, AND $v_2 \in X$, AND $v_3 \in Y$. EVIDENTLY $v_i \in X$ IFF i IS EVEN ($0 \le i \le k$). BUT

$$v_k = v_0 \in X$$

$\therefore$ k IS EVEN.

($\Leftarrow$) SUPPOSE G CONTAINS NO ODD CYCLES. ALSO ASSUME G IS CONNECTED. (IF RESULT IS PROVED for CONN. GRAPHS, IT FOLLOWS for DISCONN. GRAPHS. WHY? IF G IS DISCONN & HAS NO ODD CYCLES, THEN EACH COMPONENT HAS NO ODD CYCLES. SO EACH COMPONENT IS BIPARTITE



X

Y)

__Defn__ Given $u, v \in V(G)$

$d(u,v)$ = length of a shortest u-v PATH.

This # is well defined for all $u, v \in V(G)$ since G is CONN.

Fix $s \in V(G)$. Define

$$X = \{u \in V(G) \mid d(s,u) \text{ is even}\}$$

$$Y = \{u \in V(G) \mid d(s,u) \text{ is odd}\}$$

clearly $X \cap Y = \phi$ AND $X \cup Y = V(G)$

Claim: IF G CONTAINS AN EDGE uv with $u \in X$ & $v \in X$ or $u \in Y$ and $v \in Y$, THEN G CONTAINS AN ODD Cycle.

Note: RESULT FOLLOWS SINCE G HAS NO ODD CYCLES, SO NO SUCH EDGE EXISTS, SO X, Y is A BIPARTITION OF $V(G)$.

PROOF OF CLAIM: SUPPOSE $uv \in E(G)$ WITH $u \in X$ & $v \in X$. (THE CASE $u \in Y$ & $v \in Y$ IS SIMILAR.)

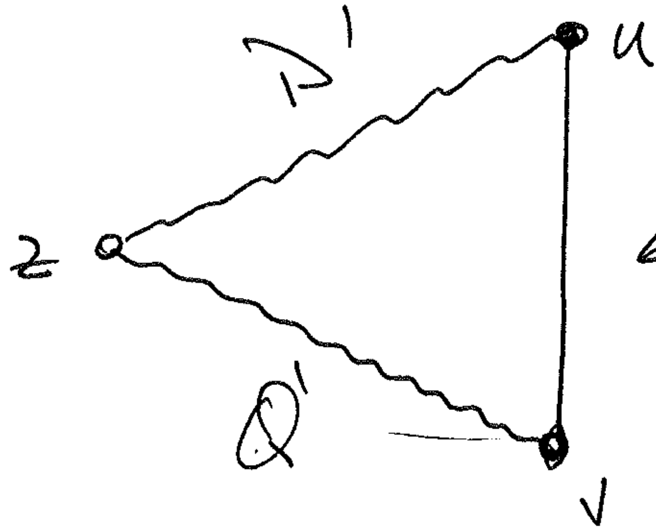
LET P BE A SHORTEST $s-u$ PATH, AND Q BE A SHORTEST $s-v$ PATH

$$P: \; s \; \underbrace{\rule{2cm}{0.4pt}}_{} \! \overset{d(s,z)}{\rule{2cm}{0.4pt}} \; z \; \overset{P'}{\rule{2cm}{0.4pt}} \; u \in X$$

$$Q: \; s \; \overset{d(s,z)}{\rule{2cm}{0.4pt}} \; z \; \overset{Q'}{\rule{2cm}{0.4pt}} \; v \in X$$

LET z BE THE LAST (FURTHEST from s) VERTEX COMMON TO P AND Q. LET P' BE THE SEGMENT OF P from z TO u, likewise for Q'.

THUS



ODD length cycle.

OBSERVE THAT length P' & length Q' HAVE SAME PARITY (i.e. BOTH even, OR BOTH odd). THUS THE cycle:

$$P' + uv + Q'$$

HAS ODD length. ///

(1.7) MATRIX REPRESENTATION

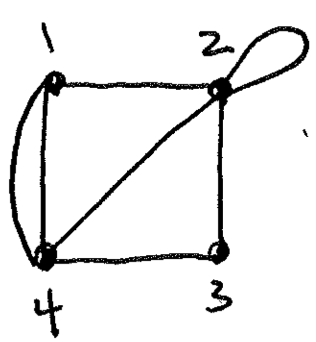
Defn: ADJACENCY MATRIX. LET
$V(G) = \{v_1, v_2, \dots, v_n\}$

$$A(G) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

WHERE a_{ij} = # of edges JOINING v_i TO v_j
$(1 \leq i \leq n,\ 1 \leq j \leq n)$

NOTE $A = A(G)$ is NECESSARILY SYMMETRIC, i.e. $a_{ij} = a_{ji}$.

EX.



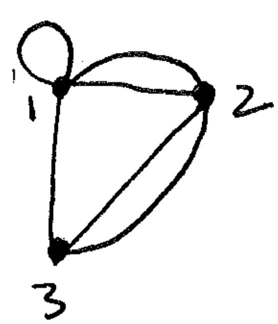
$$A = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

OBSERVE: If G has no loops, then $A(G)$ has all 0's on diagonal.

If G has no parallel edges, then $A(G)$ contains only 0's & 1's.

<u>THM</u> Let G be a graph on n vertices $V(G) = \{v_1, \ldots, v_n\}$ and let $A = A(G)$. Then for any $k \geq 0$, the ij^{th} entry in A^k is the # of v_i-v_j walks in G of length k.

<u>EX.</u>



$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

so

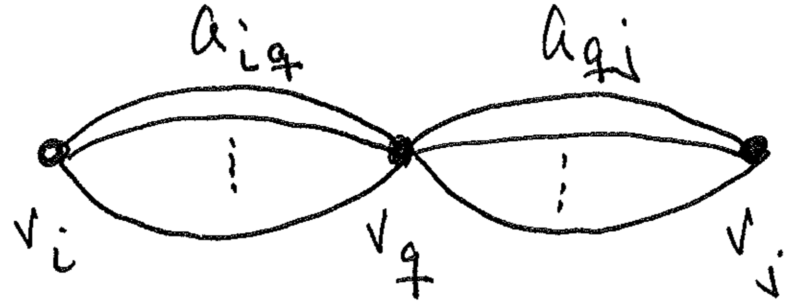
$$A^2 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 5 \\ 4 & 8 & 2 \\ 5 & 2 & 5 \end{pmatrix}$$

$$\left.\begin{matrix} (1,1,3) & \times 1 \\ (1,2,3) & \times 4 \end{matrix}\right\} = 5$$

PROOF:

Let $A = (a_{ij})$, so $a_{ij} = \#$ of edges joining v_i to v_j. Start with case $k=2$. Let $1 \le q \le n$. Observe

$a_{iq}a_{qj} = \#$ of walks of length 2 from v_i to v_j which pass through v_q.



If we sum over all possible intermediate vertices, we get total # of walks from v_i to v_j of length 2, i.e.

$$\begin{pmatrix} \text{\# of } v_i\text{-}v_j \text{ walks} \\ \text{of length } 2 \end{pmatrix} = \sum_{q=1}^{n} a_{iq} a_{qj}$$

But RHS is just the ij^{th} entry of A^2.

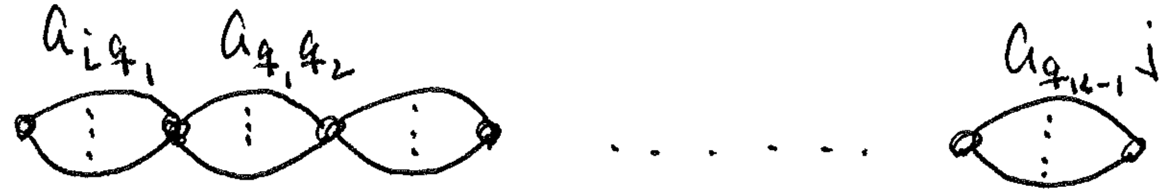
Now the general case. Consider $k-1$ vertices: $v_{q_1}, v_{q_2}, \dots, v_{q_{k-1}}$

OBSERVE that

$$a_{iq_1} a_{q_1 q_2} a_{q_2 q_3} \cdots a_{q_{k-1} j}$$

$= \#\ v_i - v_j$ walks which pass through

$v_{q_1}, v_{q_2}, \ldots, v_{q_{k-1}}$



By summing over all possible intermediate vertices $v_{q_1} \cdots v_{q_{k-1}}$, we obtain total # of $v_i - v_j$ walks of length k.

But

$$c_{ij} = \sum_{q_1} \sum_{q_2} \cdots \sum_{q_{k-1}} a_{iq_1} a_{q_1 q_2} \cdots a_{q_{k-1} j}$$

is the ij^{th} entry of A^k, i.e.

$$A^k = (c_{ij}).$$

///

<u>THM</u> let $V(G)=\{v_1,\dots,v_n\}$, AND $A=A(G)$. let

$$B = A + A^2 + \dots + A^{n-1}.$$

THEN G is CONNECTED IFF All OFF-DIAGONAL ENTRIES IN B ARE NON-ZERO. i.e. if $B=(b_{ij})$ then $i\neq j$ implies $b_{ij}\neq 0$.

<u>PROOF:</u> let $a_{ij}^{(k)}$ Denote the ij^{TH} ENTRY in A^k, which By PREV. THM is # v_i-v_j WALKS of length k. THEREFORE ij^{TH} ENTRY in B is

$$b_{ij} = a_{ij}^{(1)} + a_{ij}^{(2)} + a_{ij}^{(3)} + \dots + a_{ij}^{(n-1)}$$

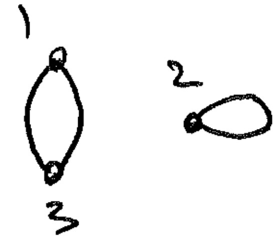
so b_{ij} = # of $v_i - v_j$ WALKS OF LENGTH less than n.

- If G is is CONNECTED, THEN for all $i \neq j$, G CONTAINS A $v_i - v_j$ PATH. SINCE G HAS n VERTICES, AND SINCE NO VERTEX is REPEATED IN A PATH, SUCH A $v_i - v_j$ PATH HAS LENGTH AT MOST $n-1$. THUS $b_{ij} \neq 0$ for all $i \neq j$.

- If $b_{ij} \neq 0$ for all $i \neq j$, THEN EACH PAIR OF DISTINCT VERTICES ARE JOINED BY A WALK (HENCE ALSO A PATH) OF length less than n. THUS v_i is REACHABLE FROM v_j for all $i \neq j$. $\therefore$ G is CONNECTED.

///.

Ex.



$n = 3$

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$B = A + A^2 = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}$$

Defn INCIDENCE MATRIX. Let

$$V(G) = \{v_1, \ldots, v_n\}$$

$$E(G) = \{e_1, \ldots, e_t\}$$

$$M = M(G) = (m_{ij}) \quad \begin{matrix} 1 \le i \le n \\ 1 \le j \le t \end{matrix}$$

m_{ij} = # times v_i is incident with e_j

i.e.

$$m_{ij} = \begin{cases} 0 & v_i \text{ not incident with } e_j \\ 1 & v_i, e_j \text{ are incident, } e_j \text{ not loop} \\ 2 & v_i, e_j \text{ are incident, } e_j \text{ is loop.} \end{cases}$$