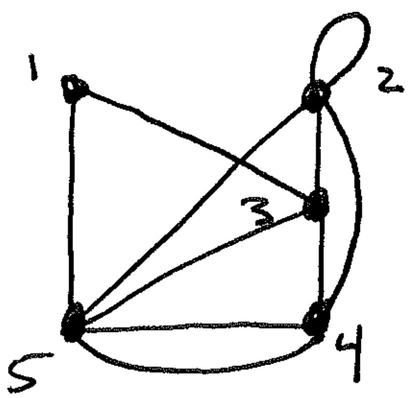
CMPE 177 6-24-09

(1.4) Degree of a vertex

Defn. Let $v \in V(G)$. The Degree $d(v)$ of v is the # of edges incident with v (loops count twice).

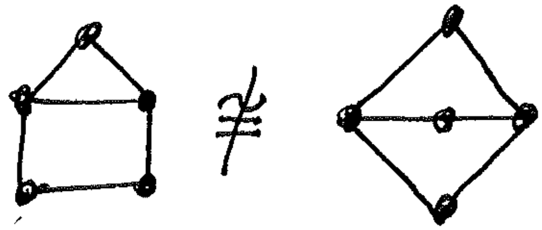
Ex.



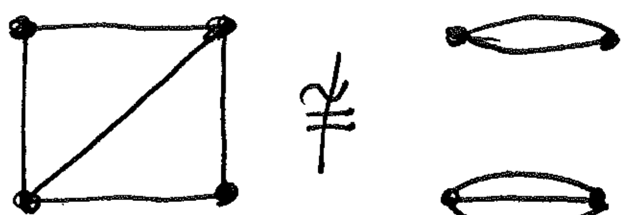
$d(1) = 2$
$d(2) = 5$
$d(3) = 4$
$d(4) = 4$
$d(5) = 5$

Defn. Degree sequence: The degrees arranged in increasing order.

Ex. 2 4 4 5 5

EX.  2 2 2 3 3

NOTATION: $G_1 \cong G_2$ SAYS 'G_1 ISOMORPHIC TO G_2'

EX  2 2 3 3

THM Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ AND $|E(G)| = e$. THEN

$$\sum_{i=1}^{n} d(v_i) = 2e$$

PROOF: EACH EDGE CONTRIBUTES 2 TO THIS SUM. ///

CALL $v \in V(G)$ <u>even</u> or <u>odd</u> ACCORDING TO whether $d(v)$ is even or odd.

<u>COROLLARY</u> ANY GRAPH HAS AN EVEN # OF ODD VERTICES.

<u>PROOF:</u> let $X \subseteq V(G)$ BE THE even VERTICES AND $Y \subseteq V(G)$ BE ODDS. MUST SHOW $|Y|$ is even. By LAST THM

$$\underbrace{2e}_{\text{even}} = \sum_{v \in V(G)} d(v) = \underbrace{\sum_{x \in X} d(x)}_{\text{even}} + \underbrace{\sum_{y \in Y} d(y)}_{\therefore\ \text{even}}$$

$\therefore \sum_{y \in Y} d(y)$ is even. $\therefore$ # OF terms in $\sum_{y \in Y} d(y)$ is even $\therefore |Y|$ is even. ///

Defn G is called k-REGULAR iff each $v \in V(G)$ has $d(v) = k$.

NOTE: if G is k-regular, $n = |V(G)|$, and $e = |E(G)|$, then

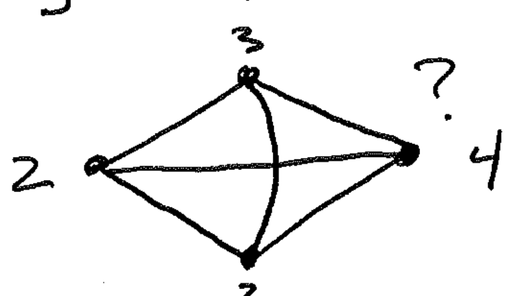
$$k \cdot n = \sum_{v \in V(G)} d(v) = 2 \cdot e$$

$$\therefore \quad e = \frac{kn}{2}$$

Ex. K_n is $(n-1)$-regular, so

$$|E(K_n)| = \frac{n(n-1)}{2}.$$

Ex. Show there is no simple graph with deg. seq. 2 3 3 4.



Thm If G is simple & $n = |V|$ then $d(v) \le n-1$.

Defn A SEQUENCE is CALLED GRAPHICAL IF IT is DEG. SEQ. OF A SIMPLE GRAPH.

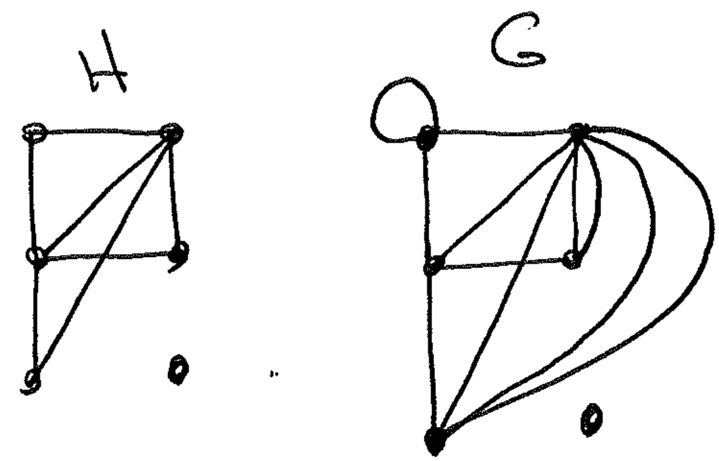
(1.5) SUBGRAPHS

Defn WE SAY A GRAPH H is A SUBGRAPH OF G IFF:

$$V(H) \subseteq V(G)$$

$$E(H) \subseteq E(G)$$

EX.



H is THE UNDERLYING SIMPLE GRAPH

H is CALLED A SPANNING SUBGRAPH IFF $V(H) = V(G)$

If $e \in E(G)$ we denote by $G-e$ the subgraph of G obtained by deleting e.

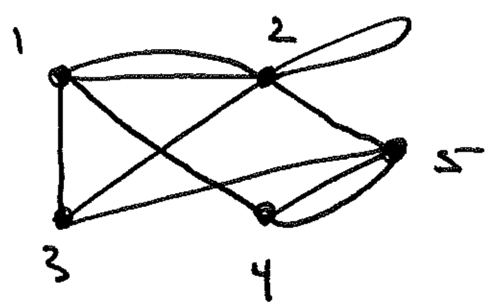
If $F \subseteq E(G)$, then $G-F$ denotes the subgraph obtained by deleting all $e \in F$.

If $v \in V(G)$ then $G-v$ denotes subgraph obtained by deleting v and all incident edges.

If $U \subseteq V(G)$, then $G-U$ denotes result of deleting all $v \in U$, along with incident edges.

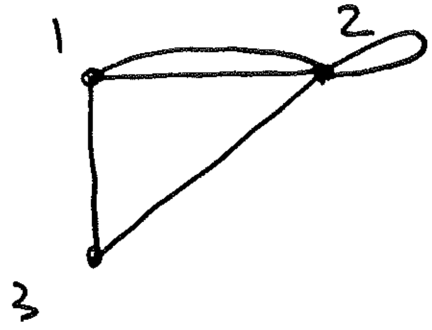
<u>DEFN</u>. let $U \subseteq V(G)$. The <u>SUBGRAPH INDUCED BY U</u>, DENOTED $G[U]$, is THE GRAPH WITH VERTEX SET U, AND EDGE SET CONSISTING OF ALL EDGES OF G WITH <u>BOTH</u> ends in U.

<u>EX.</u>



G $V = \{1, 2, 3, 4, 5\}$

$E = \{ \ldots \}$



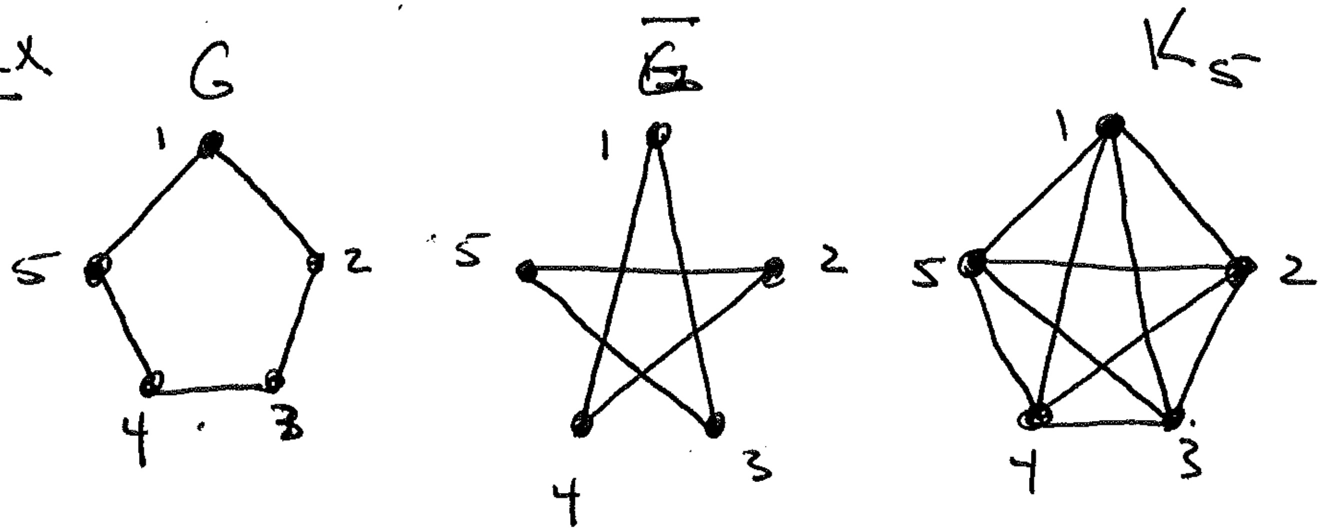
$G[\{1, 2, 3\}]$

CAN SIMILARLY DEFINE THE SUBGRAPH INDUCED BY $F \subseteq E(G)$, DENOTED $G[F]$.

Defn

LET G BE A SIMPLE GRAPH. THE COMPLEMENT OF G, DENOTED $\overline{G}$, IS THE GRAPH WITH $V(\overline{G}) = V(G)$, SUCH THAT $u, v \in V(\overline{G})$ ARE ADJACENT IN $\overline{G}$ IFF THEY ARE NOT ADJACENT IN G.

Ex



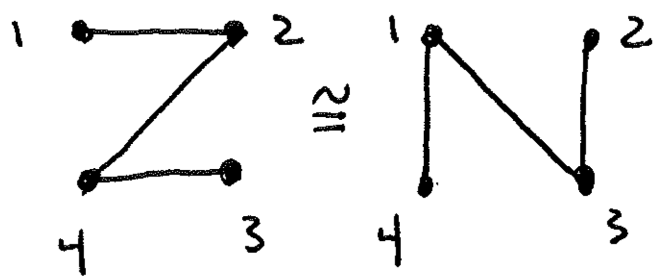
NOTICE: $E(G) \cap E(\overline{G}) = \emptyset$ AND

$E(G) \cup E(\overline{G}) = E(K_n)$, $n = |V(G)|$.

Defn

A SIMPLE GRAPH G IS CALLED SELF-COMPLEMENTARY IFF $G \cong \overline{G}$.

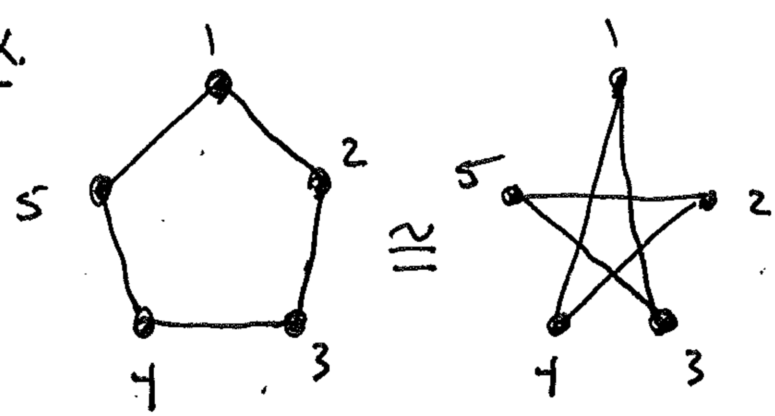
EX.



ISOMORPHISM

$1 \to 2$
$2 \to 3$
$3 \to 4$
$4 \to 1$

EX.



ISOMORPHISM

$1 \to 1$
$2 \to 3$
$3 \to 5$
$4 \to 2$
$5 \to 4$

<u>PROB 1.5.3 (P.23)</u> ASK TO PROVE:

If G is SELF-COMPLEMENTARY, THEN EITHER $n \equiv 0 \pmod 4$ OR $n \equiv 1 \pmod 4$, where $n = |V(G)|$.

<u>Defn</u>. $x \equiv y \pmod m$ IFF m EVENLY DIVIDES $(x-y)$. EQUIVALENTLY x & y HAVE SAME REMAINDER UPON DIVISION BY m.

<u>Ex.</u> $10 \equiv 14 \pmod 4$

IN OTHER WORDS EITHER $n = 4t$ OR $n = 4t+1$, for some integer t.

IN PARTICULAR THERE CAN BE NO SELF-COMPLEMENTARY GRAPH WITH 6 OR 7 VERTICES.

SO POSSIBLE # OF VERTICES ~~ON~~ IN A SELF-COMP. GRAPH ARE

$$1, 4, 5, 8, 9, 12, 13, \ldots\ldots$$

EXERCISE!

FIND A SIMPLE GRAPH ON 8 VERTICES WHICH IS SELF-COMPLEMENTARY

(1.6) PATHS & CYCLES

Defn A WALK in G is AN ALTERNATING SEQUENCE OF VERTICES & EDGES OF THE FORM

$$v_0\, e_1\, v_1\, e_2\, v_2 \cdots e_{k-1}\, v_{k-1}\, e_k\, v_k$$

SUCH THAT e_i JOINS v_{i-1} TO v_i $(1 \leq i \leq k)$. ALSO CALLED A $v_0 - v_k$ WALK.

NOTE: IF G IS SIMPLE ITS SUFF. TO LIST ONLY THE VERTICES.

NOTE: VERTICES & EDGES IN A WALK NEED NOT BE DISTINCT.

IF $v_0 = v_k$ THE WALK IS SAID TO BE CLOSED.

Defn A walk in which all edges are distinct is called a trail.

Defn A trail in which all vertices are distinct (except possibly $v_0 = v_k$) is called a path.

Defn. A closed path is called a cycle.

Defn The length of a walk is the # of edge traversals.

NOTE: Given any u-v walk, we can delete some vertices & edges to obtain a u-v path.

<u>DEFN</u>

WE SAY $v \in V(G)$ IS <u>REACHABLE</u> FROM $u \in V(G)$ IFF G CONTAINS A u-v PATH (OR u-v WALK).

<u>DEFN</u> G IS SAID TO BE <u>CONNECTED</u> IFF FOR $u, v \in V(G)$ v IS REACHABLE FROM u. OTHERWISE G IS CALLED <u>DISCONNECTED</u>.