CMPE 177 GRAPH THEORY & ALGORITHMS

6-22-09

(1.1) DEFINITIONS.

<u>DEFN</u>: A <u>GRAPH</u> $G = (V, E)$ IS A PAIR OF FINITE SETS.

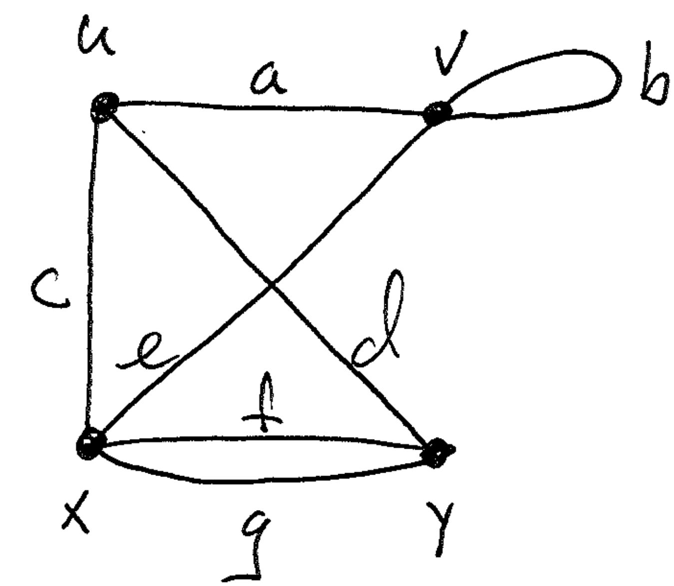
(1) $V = V(G) \neq \emptyset$ THE <u>VERTEX</u> SET.

(2) $E = E(G)$ (POSSIBLY EMPTY) THE <u>EDGE</u> SET. ELEMENTS OF E CORRESPOND TO <u>UNORDERED PAIRS</u> OF VERTICES.

EX. $V = \{u, v, x, y\}$ $E = \{a, b, c, d, e, f\}$

$$\left.\begin{array}{l}
a \Leftrightarrow (u,v) = (v,u) \\
b \Leftrightarrow (v,v) \quad \underline{\text{loop}} \\
c \Leftrightarrow (u,x) \\
d \Leftrightarrow (u,y) \\
e \Leftrightarrow (x,v) \\
\end{array}\right.$$

$$\left.\begin{array}{l}
f \Leftrightarrow (x,y) \\
g \Leftrightarrow (x,y)
\end{array}\right\} \text{MULTIPLE OR PARALLEL EDGES}$$



u is ADJACENT TO v

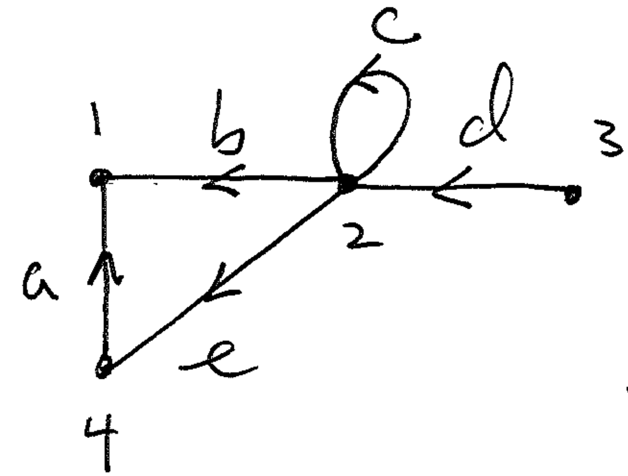
a is ADJACENT TO c

u is INCIDENT WITH a

Defn: **SIMPLE** GRAPH: NO LOOPS OR PARALLEL EDGES.

DEFN: **DIRECTED GRAPH** (**DIGRAPH**): ELEMENTS OF E CORRESPOND TO **ORDERED** PAIRS OF VERTICES.

EX.



$V = \{1, 2, 3, 4\}$

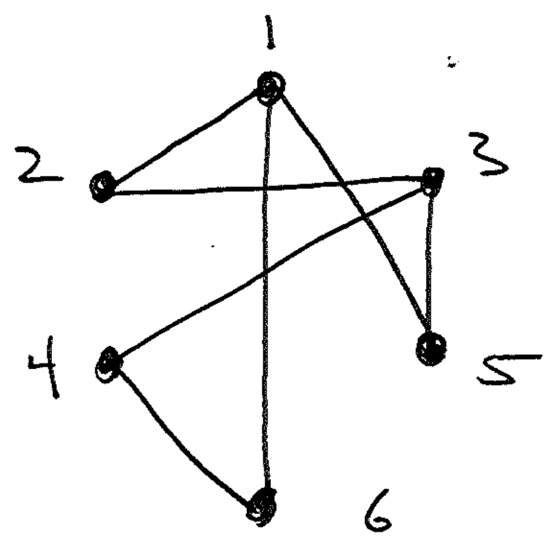
$E = \{a, b, c, d, e\}$

$a \Leftrightarrow (4, 1) \neq (1, 4)$

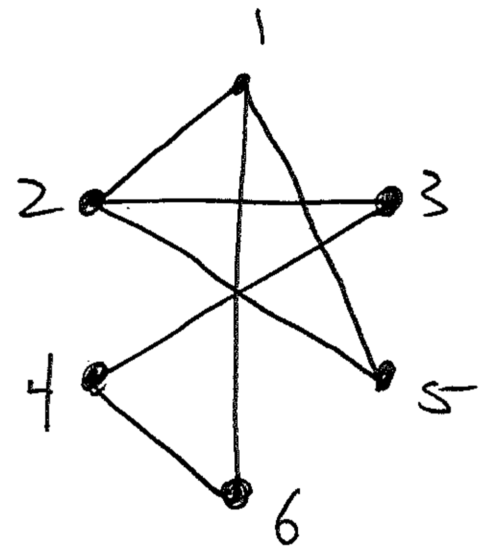
ORIGIN TERMINUS

<u>EX.</u> <u>HAMILTONIAN CYCLE</u> PROBLEM

GIVEN V A SET OF CITIES & E A SET OF HIGHWAYS, DOES THERE EXIST A ROUND TRIP ALONG WHICH A SALESMAN CAN TRAVEL TO VISIT EACH CITY EXACTLY ONCE ?

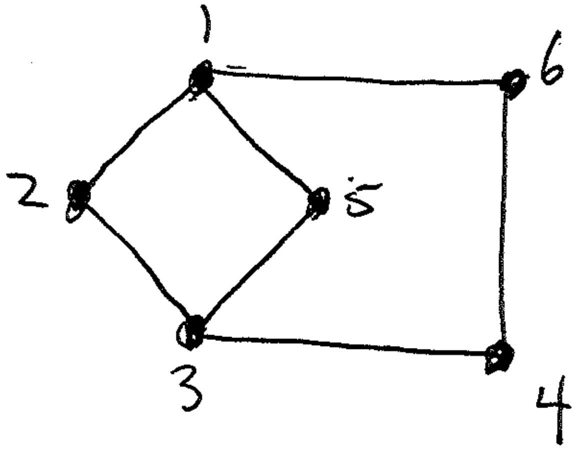


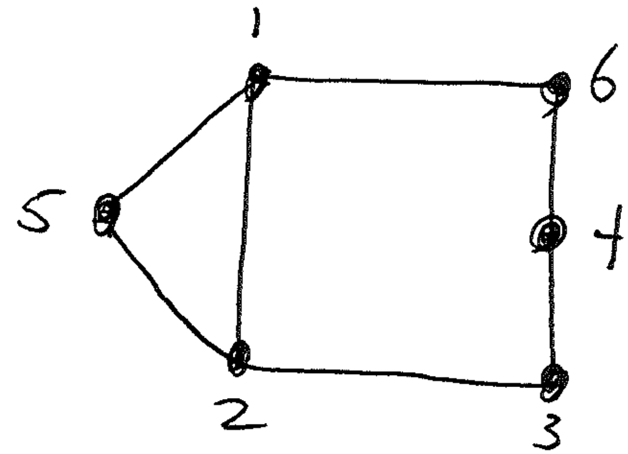
<u>NO</u>



<u>YES</u>

Redraw:





CYCLE: 2, 5, 1, 6, 4, 3, 2

NON-HAMILTONIAN

HAMILTONIAN

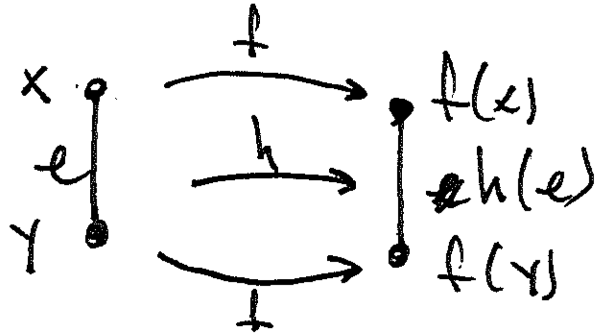
(1.3) MORE DEFNS.

DEFN: TWO GRAPHS $G_1 = (V_1, E_1)$ AND $G_2 = (V_2, E_2)$ ARE SAID TO BE ISOMORPHIC IFF THERE EXIST BIJECTIONS

$$f : V_1 \to V_2 \quad \& \quad h : E_1 \to E_2$$

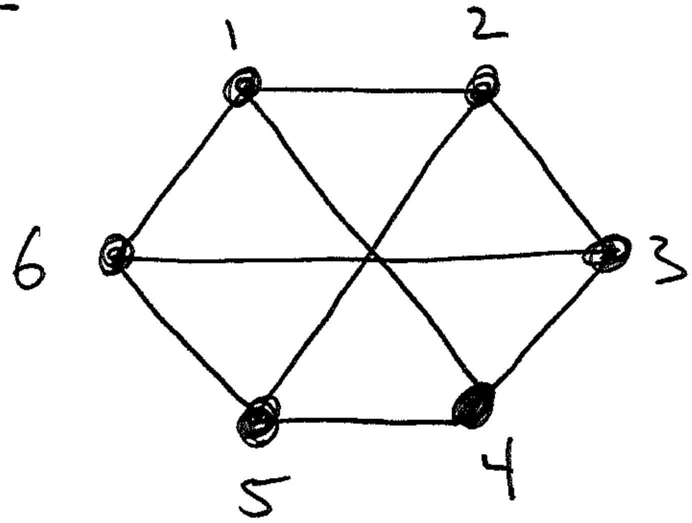
SATISFYING :

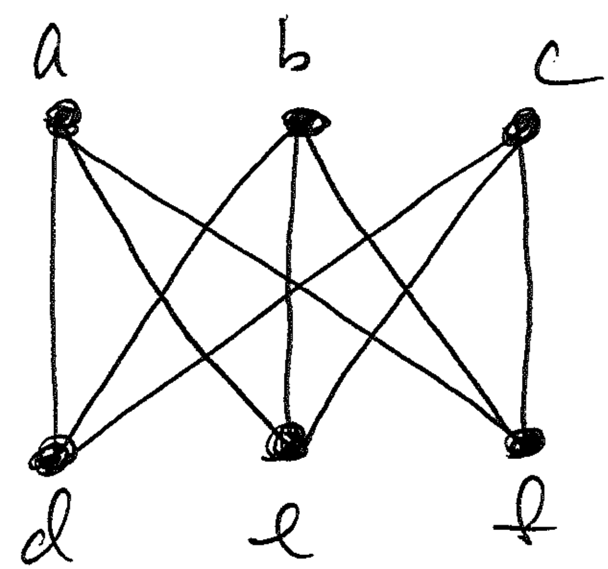
$e \in E_1$ JOINS $x, y \in V_1$ IFF
$h(e) \in E_2$ JOINS $f(x), f(y) \in V_2$



THE PAIR (f, h) IS CALLED AN <u>ISOMORPHISM</u>

<u>EX</u>





<u>SHORTHAND</u>

$12 = (1, 2)$

<u>ISOMORPHISM</u>:

$$V: \left\{ \begin{array}{l} 1 \to a \\ 2 \to d \\ 3 \to b \\ 4 \to e \\ 5 \to c \\ 6 \to f \end{array} \right. \qquad E: \left\{ \begin{array}{ll} 12 \to ad & 34 \to be \\ 14 \to ae & 36 \to bf \\ 16 \to af & 45 \to ec \\ 23 \to db & 56 \to cf \\ 25 \to dc & \end{array} \right.$$

ANOTHER ISOMORPHISM

$$V : \begin{cases} 1 \to b \\ 2 \to d \\ 3 \to c \\ 4 \to f \\ 5 \to a \\ 6 \to e \end{cases} \qquad E : \begin{cases} \ldots \text{EXERCISE} \ldots \end{cases}$$

EXERCISE:

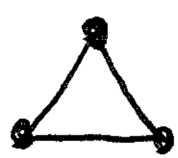
- WRITE DOWN A FEW MORE ISOMORPHISMS
- HOW MANY ISOMORPHISMS ARE THERE?
  ANSWER: 72

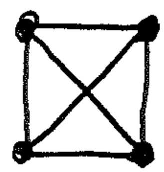
<u>DEFN</u> <u>COMPLETE GRAPH</u> : A SIMPLE GRAPH IN WHICH EACH PAIR OF DISTINCT VERTICES IS JOINED BY AN EDGE.

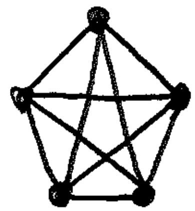
- THERE IS ONLY ONE COMPLETE GRAPH ON n VERTICES (UP TO ISOMORPHISM) CALLED K_n

K_1  0 EDGES

K_2  1 EDGE

K_3  3 EDGES

K_4  6 EDGES

K_5  10 EDGES

⋮ EXERCISE: CONTINUE

EXERCISE: HOW MANY EDGES HAS K_n?

ANSWER: $|E(K_n)| = \frac{n(n-1)}{2}$

- RECALL **BINOMIAL COEFFICIENTS**

$\binom{n}{m}$ = # of m-SUBSETS IN AN n-SET

$0 \leq m \leq n$

THM: $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$

$|E(K_n)|$ = # 2-SUBSETS of $V(K_n)$

$$= \binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)\cancel{(n-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{2\cdot 1\cdot\cancel{(n-2)}\cancel{(n-3)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$$= \frac{n(n-1)}{2}.$$

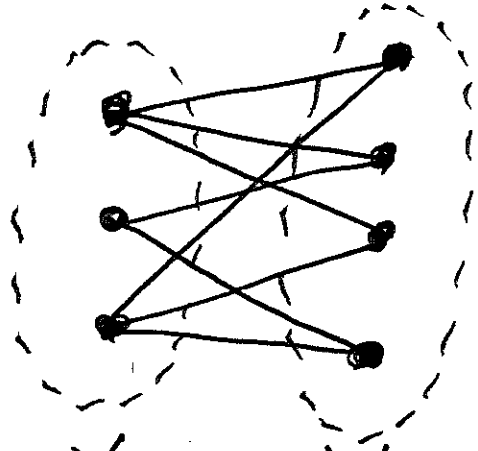
<u>Defn:</u> A <u>Null</u> Graph has $E = \phi$.

Notation: N_n null graph on n vertices.

<u>Defn:</u> <u>Bipartite</u> Graph: $V(G)$ can be partitioned into two (non-empty) sets X, Y (i.e. $X \cap Y = \phi$, $X \cup Y = V(G)$.) such that each edge has one end in X and one end in Y (i.e. no edge joins X to X or Y to Y.)

The pair X, Y is called a <u>Bipartition</u>

<u>EX.</u>



C.B.G.

<u>Defn</u>: <u>COMPLETE BIPARTITE GRAPH</u> : is A SIMPLE BIPARTITE GRAPH IN WHICH EACH VERTEX IN X IS JOINED TO EACH VERTEX IN Y.

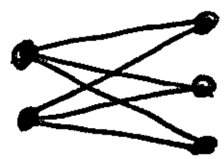
<u>FACT</u>: UP TO ISOMORPHISM THERE IS ONLY ONE CBG WITH $|X| = n$, $|Y| = m$

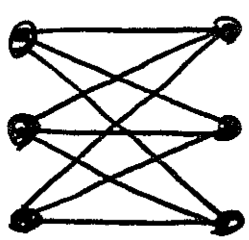
NOTATION: $K_{n,m}$

$K_{1,1}$ 

$K_{1,2}$ 

$K_{2,2}$ 

$K_{2,3}$ 

$K_{3,3}$ 

NOTICE!

$|E(K_{n,m})| = n \cdot m$

HW1 : DUE NEXT MON.

P. 13 : #2

P. 15 : #3, 4, 5, 11

P. 23 : #3

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