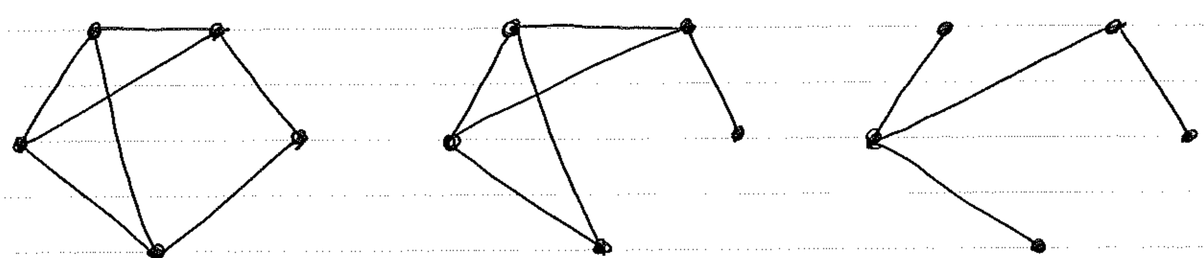
3.3 Hamiltonian Graphs

Let G be a connected graph. A Hamiltonian path in G is simply a path which includes every vertex of G. A Hamiltonian cycle is a closed Hamiltonian path.

G is called Hamiltonian if it contains a Hamiltonian cycle.

A non-Hamiltonian graph is called semi-Hamiltonian if it contains a (necessarily non-closed) Hamiltonian path.

Ex.



Hamiltonian — Semi-Hamiltonian — Neither

Observe that if you insert enough new edges into a non-Hamiltonian graph it becomes Hamiltonian.

Note: G is Hamiltonian iff its underlying simple graph is Hamiltonian. (Why?)

∴ we usually assume G is simple.

UNFORTUNATELY THERE IS NO (KNOWN) SIMPLE CRITERION FOR DECIDING WHEN A GRAPH IS OR IS NOT HAMILTONIAN (AS THERE WAS FOR EULERIAN GRAPHS.)

FINDING A NECESSARY AND SUFFICIENT CONDITION (i.e. IF AND ONLY IF) FOR A GRAPH TO BE HAMILTONIAN IS AN UNSOLVED PROBLEM IN GRAPH THEORY.

SUFFICIENT CONDITIONS (i.e. IF - THEN) ARE GENERALLY OF THE FORM: IF G HAS "ENOUGH" EDGES, THEN G IS HAMILTONIAN

<u>THEOREM</u>: (ORE)
LET G BE A SIMPLE GRAPH ON $n \geq 3$ VERTICIES, AND SUPPOSE THAT FOR EACH PAIR OF NON-ADJACENT VERTICIES x, y IN G WE HAVE

$$\deg(x) + \deg(y) \geq n .$$

THEN G IS HAMILTONIAN.

ANOTHER RESULT ALONG THESE LINES IS ATTRIBUTED TO DIRAC.

COROLLARY (DIRAC)

LET G BE A SIMPLE GRAPH WITH $n \geq 3$ VERTICIES, AND SUPPOSE THAT FOR EACH VERTEX x, $\deg(x) \geq \frac{n}{2}$. THEN G IS HAMILTONIAN.

PROOF:

THIS FOLLOWS IMMEDIATELY FROM ORE'S THEOREM SINCE FOR ANY PAIR OF VERTICIES x, y (ADJACENT OR NOT) WE HAVE

$$\deg(x) + \deg(y) \geq \frac{n}{2} + \frac{n}{2} = n. \qquad ///.$$

PROOF OF ORE'S THM:

LET G BE A SIMPLE GRAPH ON $n \geq 3$ VERTICIES SATISFYING THE GIVEN CONDITION ON THE VERTEX DEGREES. ASSUME (TO GET A CONTRADICTION) THAT G IS NON-HAMILTONIAN.

IF NECESSARY ADD EDGES TO G IN SUCH A WAY THAT IT REMAINS SIMPLE AND NON-HAMILTONIAN, UNTIL THE ADDITION OF JUST ONE MORE EDGE WOULD CREATE A HAMILTONIAN CYCLE. NOTE THAT IN THIS NEW GRAPH THE VERTEX DEGREE CONDITION IS STILL SATISFIED. CALL THIS GRAPH G'.

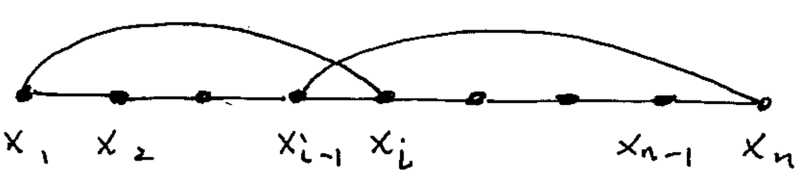
G' MUST THEN BE SEMI-HAMILTONIAN, i.e. THERE IS A PATH $x_1 \to x_2 \to \cdots \to x_n$ PASSING THROUGH EVERY VERTEX. BUT SINCE G' IS NOT HAMILTONIAN, x_1 AND x_n ARE NOT ADJACENT, WHENCE

$$\deg(x_1) + \deg(x_n) \geq n,$$

NOTE THAT IN A SIMPLE GRAPH $\deg(x)$ IS THE NUMBER OF VERTICIES ADJACENT TO x. THUS THERE ARE $\deg(x_1)$ MANY VERTICIES x_i ADJACENT TO x_1, AND EACH HAS A PREDECESSOR x_{i-1}. IF NONE OF THESE PREDECESSORS WERE ADJACENT TO x_n, THEN x_n WOULD BE ADJACENT TO AT MOST $n-1-\deg(x_1)$ VERTICIES. i.e. $\deg(x_n) \leq n-1-\deg(x_1)$, WHENCE

$$\deg(x_1) + \deg(x_n) \leq n-1$$

CONTRADICTING THE PREVIOUS INEQUALITY. THUS THERE EXISTS A CONSECUTIVE PAIR x_{i-1}, x_i IN THE SEQUENCE SUCH THAT x_i IS ADJACENT TO x_1, AND x_{i-1} IS ADJACENT TO x_n.



BUT THEN

$$x_1 \Rightarrow x_2 \Rightarrow \cdots \Rightarrow x_{i-1} \Rightarrow x_n \Rightarrow x_{n-1} \Rightarrow \cdots \Rightarrow x_i \Rightarrow x_1$$

IS A HAMILTONIAN CYCLE IN G'.
THIS CONTRADICTION SHOWS THAT OUR ORIGINAL GRAPH G MUST HAVE ITSELF BEEN HAMILTONIAN.

///.

THEOREM.
LET G BE SIMPLE. LET u, v BE NON-ADJACENT VERTICES IN G SUCH THAT

$$d(u) + d(v) \geq n.$$

LET $G + uv$ DENOTE THE GRAPH OBTAINED BY JOINING u TO v BY A NEW EDGE. THEN G IS HAMILTONIAN IF AND ONLY IF $G + uv$ IS.

PROOF.
($\Rightarrow$) IF G CONTAINS A HAMILTONIAN CYCLE, THEN OBVIOUSLY THE SUPERGRAPH $G + uv$ CONTAINS ONE ALSO.

($\Leftarrow$) WE PROCEED IN A WAY SIMILAR TO THE PROOF OF ORE'S THEOREM.

SUPPOSE $G+uv$ IS HAMILTONIAN. ASSUME, TO GET A CONTRADICTION, THAT G IS NOT. THEN IN FACT G MUST CONTAIN A SEMI-HAMILTONIAN u-v PATH:

$$u = x_1 \to x_2 \to \cdots\cdots \to x_{n-1} \to x_n = v$$

JUST AS BEFORE, IF x_n IS NOT ADJACENT TO A PREDECESSOR x_{i-1} OF A NEIGHBOR x_i OF x_1, THEN

$$d(x_1) + d(x_n) \leq n-1,$$

CONTRARY TO HYPOTHESIS.

THUS THERE IS A CONSECUTIVE PAIR x_{i-1}, x_i IN THIS PATH ADJACENT TO x_n AND x_1 RESPECTIVELY, AND AS BEFORE

$$x_1 \to x_2 \to \cdots \to x_{i-1} \to x_n \to x_{n-1} \to \cdots \to x_i \to x_1$$

IS A HAMILTONIAN CYCLE IN G, A CONTRADICTION. THEREFORE G MUST BE HAMILTONIAN.

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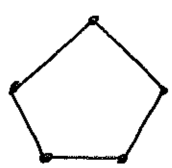
DEFN.

LET G BE A SIMPLE GRAPH ON n VERTICES. THE <u>CLOSURE</u> OF G, DENOTED $c(G)$ IS THE SUPERGRAPH OBTAINED AS FOLLOWS:

1.) while THERE EXIST NON-ADJACENT $u, v \in V(G)$ SUCH THAT $d(u) + d(v) \geq n$

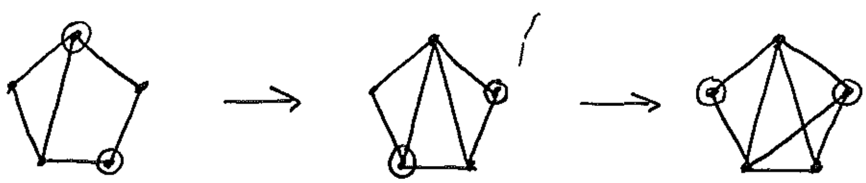
2.) JOIN u TO v BY A NEW EDGE.

<u>EXAMPLES</u> $n = 5$

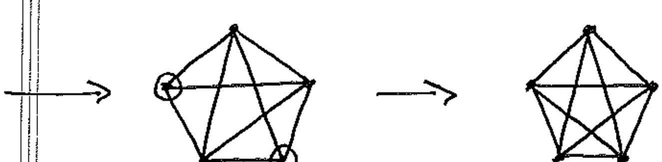
(a)  $G = c(G)$

(b) 

G c(G)

(c) 

G



c(G)

Corollary

A simple graph G is Hamiltonian if and only if $c(G)$ is Hamiltonian

Proof.

Let $G = G_1, G_2, \ldots, G_k = c(G)$ be the sequence of obtained by the above algorithm. By the preceding theorem

$G = G_1$ is Hamiltonian
iff G_2 is Hamiltonian
⋮
iff $G_k = c(G)$ is Hamiltonian.

///.

Corollary

Let G be a simple graph on $n \geq 3$ vertices. If $c(G) = K_n$, then G is Hamiltonian.

Proof:

K_n is Hamiltonian for $n \geq 3$. ///

Note that the converse is false by example (a) above.