3.1 EULERIAN GRAPHS

RECALL DEFINITIONS OF WALK, TRAIL, PATH, CYCLE.

AN EULER TRAIL IS SIMPLY A TRAIL WHICH INCLUDES ALL EDGES IN G.

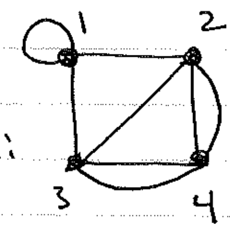
AN EULER TOUR (OR EULER CIRCUIT) IS A CLOSED EULER TRAIL.

A CONNECTED GRAPH G IS CALLED EULERIAN IF IT CONTAINS AN EULER TOUR.

G IS CALLED SEMI-EULERIAN IF IT CONTAINS A NON-CLOSED EULER TRAIL.

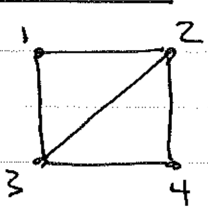
EX.

EULERIAN:



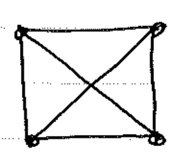
$1 \to 2 \to 3 \to 4 \to 2 \to 4 \to 3 \to 1 \to 1$

SEMI-EULERIAN

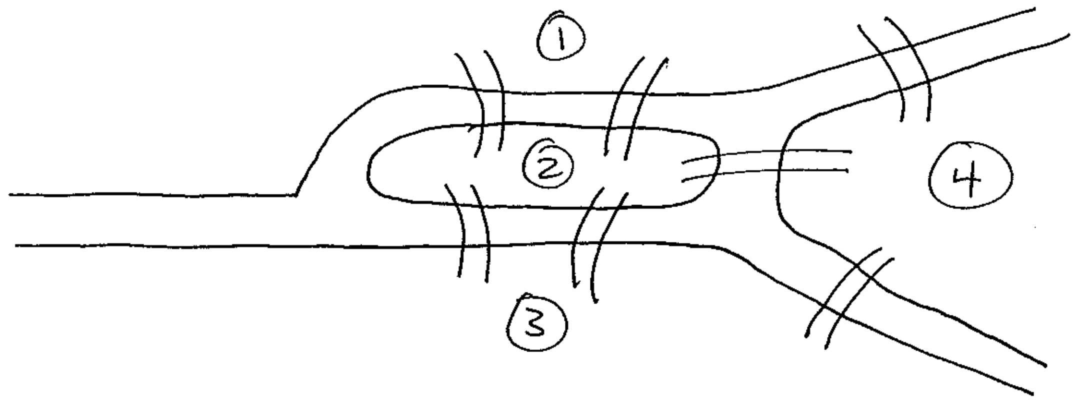


$2 \to 1 \to 3 \to 2 \to 4 \to 3$

NON-EULERIAN, NON-SEMI-EULERIAN

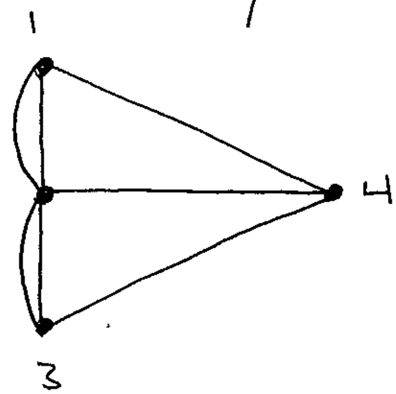


KÖNIGSBERG BRIDGE PROBLEM:



IS THERE A CLOSED TOUR OF THE CITY WHICH CROSSES EACH OF THE SEVEN BRIDGES EXACTLY ONCE ?

G:



is G EULERIAN?

ANSWER: NO . G is ALSO NOT SEMI-EULERIAN, SO THERE is NO TOUR, CLOSED OR NOT WHICH CROSSES EACH BRIDGE JUST ONCE.

THEOREM (EULER)
A CONNECTED GRAPH is EULERIAN IF AND ONLY IF EACH VERTEX is OF EVEN DEGREE.

TO PROVE THIS WE FIRST NEED

LEMMA:

IF THE VERTICIES OF G EACH HAVE DEGREE AT LEAST 2, THEN G CONTAINS A CYCLE.

PROOF:

WE ASSUME G IS SIMPLE (OTHERWISE THE CONCLUSION IS TRIVIALLY TRUE.) LET x BE ANY VERTEX OF G AND LET x_1 BE ADJACENT TO x. CHOOSE x_2 TO BE ANY VERTEX ADJACENT TO x_1, OTHER THAN x. (THIS IS POSSIBLE BY OUR HYPOTHESIS.) PICK $x_3, x_4, \ldots$ SIMILARLY, AND IN GENERAL CHOOSE x_{i+1} TO BE ANY VERTEX ADJACENT TO x_i, OTHER THAN x_{i-1}.

EXTEND THE PATH $x \Rightarrow x_1 \Rightarrow x_2 \Rightarrow \cdots \Rightarrow x_i \Rightarrow \cdots$ UNTIL SOME VERTEX x_k IS VISITED TWICE. (THIS MUST HAPPEN SINCE THERE ARE ONLY FINITELY MANY SUCH VERTICIES.) THE PART OF THE PATH STARTING AND STOPING AT x_k IS THEN THE REQUIRED CYCLE:

$$x_k \Rightarrow x_{k+1} \Rightarrow \cdots \Rightarrow x_k \quad .$$

PROOF: (OF EULER'S THEOREM)

($\Rightarrow$) LET P BE AN EULERIAN TRAIL IN G. WHEN P PASSES THROUGH A VERTEX, IT CONTRIBUTES 2 TO ITS DEGREE. SINCE EACH EDGE IS TRAVERSED EXACTLY ONCE, THE DEGREE OF EACH VERTEX IS THE SUM OF THESE CONTRIBUTIONS, WHICH IS THEREFORE EVEN.

($\Leftarrow$) WE USE INDUCTION ON THE NUMBER OF EDGES. SUPPOSE THE VERTEX DEGREES OF G ARE ALL EVEN, AND ASSUME (AS INDUCTIVE HYPOTHESIS) THAT THE THEOREM IS TRUE FOR ANY GRAPH WITH FEWER EDGES THAN G. SINCE G IS CONNECTED, NO VERTEX HAS DEGREE ZERO, SO ALL VERTEX DEGREES ARE AT LEAST TWO. BY THE PREVIOUS LEMMA, G CONTAINS A CYCLE, CALL IT C.

LET H BE THE SUBGRAPH OF G OBTAINED BY REMOVING THE EDGES OF C. H MAY BE DISCONNECTED; SAY ITS COMPONENTS ARE

$$H = H_1 \cup H_2 \cup \cdots \cup H_k$$

NOW EACH COMPONENT H_i IS CONNECTED, HAS FEWER EDGES THAN G, AND HAS ONLY EVEN VERTEX DEGREES (WHY?). WE MAY THEREFORE APPLY THE INDUCTION HYPOTHESIS TO OBTAIN AN EULERIAN TRAIL P_i IN EACH H_i $(i = 1, 2, \ldots, k)$.

WE CAN NOW CONSTRUCT AN EULERIAN TRAIL IN G AS FOLLOWS. START ON ANY VERTEX OF C, AND FOLLOW THE EDGES OF C UNTIL A NON-ISOLATED VERTEX OF H IS REACHED, SAY IN COMPONENT H_i, DEPART FROM C AND TRAVERSE THE EULERIAN TRAIL P_i, RETURNING TO THE SAME VERTEX OF C, CONTINUE ALONG C TO THE NEXT NON-ISOLATED VERTEX OF H, AND TRAVERSE THE EULERIAN TRAIL IN ITS COMPONENT.

CONTINUING IN THIS WAY WE EVENTUALLY RETURN TO OUR STARTING POINT ON C, HAVING TRAVERSED EACH EDGE OF G EXACTLY ONCE, THEREFORE G IS EULERIAN.

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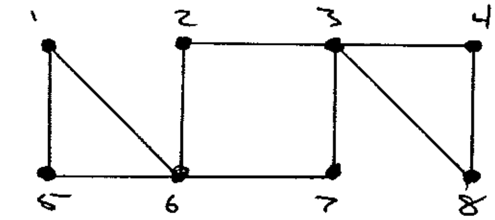
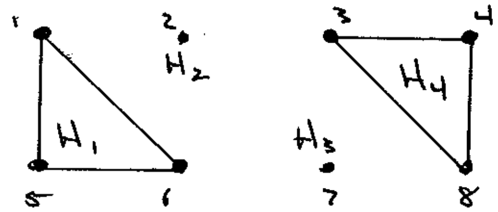
Ex. (illustrate proof.)

G:



C: $2 \to 3 \to 7 \to 6 \to 2$

H:



P_1: $6 \to 5 \to 1 \to 6$
P_4: $3 \to 4 \to 8 \to 3$

D: $2 \to P_4 \to 7 \to P_1 \to 2$

Corollary:
A connected graph is Eulerian if and only if its edge collection can be partitioned into disjoint cycles. (See example.)

Proof:
By the above theorem, it is equivalent to show that for any connected graph, its edge set partitions into disjoint cycles if and only if each vertex degree is even.

($\Rightarrow$) Suppose $E(G)$ partitions into (disjoint) cycles. Each cycle contributes 2 to the degrees of each of its vertices. Since each edge of G belongs to exactly one such cycle, the degree of each vertex is the sum of these contributions, hence is even.

($\Leftarrow$) Suppose each vertex of G has even degree. We proceed again by induction on $|E(G)|$.

<u>Base</u>: if $|E(G)| = 1$ then G must be  and $E(G)$ consists of exactly one cycle.

<u>Induction</u>: Let $|E(G)| > 1$ and assume the result holds for all graphs with fewer edges. i.e. we assume

IND. HYP. $\left\{\begin{array}{l}\text{if } H \text{ is connected with all even degrees,} \\ \text{and } |E(H)| < |E(G)|, \text{ then } E(H) \\ \text{partitions into (disjoint) cycles.}\end{array}\right.$

As before G is connected, $\therefore$ has no zero degrees, $\therefore$ all degrees are at least 2, $\therefore$ G contains a cycle C.

LET $H_1, H_2, \ldots, H_k$ BE THE COMPONENTS OF $G - C = (V(G), E(G) - E(C))$.

EACH GRAPH H_i IS CONNECTED, HAS FEWER EDGES THAN G, AND HAS ALL EVEN VERTEX DEGREES (WHY?)

$\therefore$ EACH $E(H_i)$ PARTITIONS INTO CYCLES, AND SO ALSO DOES

$$E(G) = E(C) \cup E(H_1) \cup \cdots \cup E(H_k).$$

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COROLLARY

A CONNECTED GRAPH IS SEMI-EULERIAN IF AND ONLY IF IT HAS EXACTLY TWO VERTICES OF ODD DEGREE.

EXCERCISE

PROVE THIS. ($\Rightarrow$ DIRECTLY. $\Leftarrow$ INDUCTION ON THE NUMBER OF EDGES.)

NOTE: EULERIAN GRAPHS ARE IN SOME SENSE THE OPPOSITE OF TREES SINCE ALL EDGES ARE CYCLE (i.e. NON-BRIDGE) EDGES. BUT NOTE THAT "ALL CYCLE EDGES" $\not\Leftrightarrow$ "E PARTITIONS INTO CYCLES."

EX.

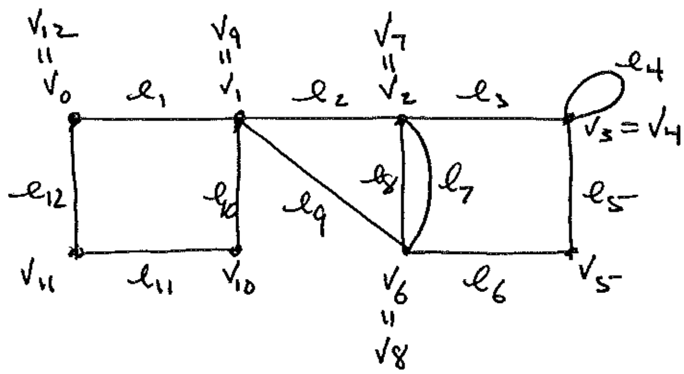
GIVEN THAT G IS EULERIAN, THE FOLLOWING ALGORITHM PRODUCES AN EULERIAN TRAIL IN G: $v_0 e_1 v_1 e_2 \cdots e_{m-1} v_{m-1} e_m v_0$.

FLEURY'S ALGORITHM

1.) PICK $v_0 \in V(G)$
2.) FOR $i \leftarrow 1$ TO $m = |E(G)|$
3.) PICK $e_i \in G - \{e_1, \ldots, e_{i-1}\}$ SUCH THAT
 (i) e_i IS INCIDENT WITH v_{i-1}
 (ii) IF POSSIBLE, e_i IS A NON-BRIDGE IN THE GRAPH $G - \{e_1, \ldots, e_{i-1}\}$.
4.) $v_i \leftarrow$ THE "OTHER" END OF e_i.

i.e. ONLY TRAVERSE A BRIDGE WHEN YOU HAVE NO CHOICE.

EX.



READ PROOF OF VALIDITY OF FLEURY. (THM 3.5. P. 91.)

READ CHINESE POSTMAN PROBLEM 3.2 (P. 96)