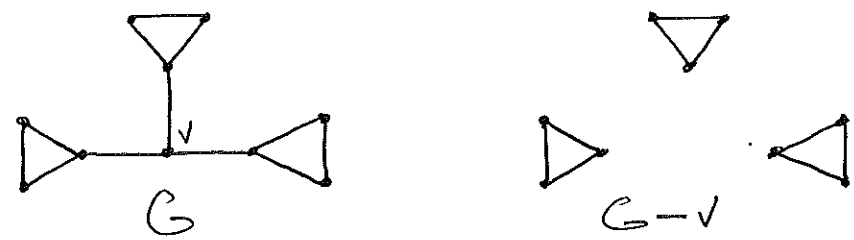
2.6 Cut Vertices

Defn.
We call $v \in V(G)$ a cut vertex (or articulation point) if $\omega(G-v) > \omega(G)$.

Ex



Theorem.
Let G be connected. $v \in V(G)$ is a cut vertex if and only if there exist vertices u and w (other than v) such that v lies on every u-w path.

Rmk: The result also holds for disconnected graphs. Just apply the theorem to $Comp_G(v)$.

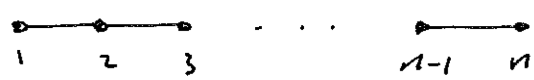
Proof:

($\Rightarrow$) Suppose v is a cut vertex in G. Let u and w be chosen from different components of $G-v$. Although G contains a u-w path, $G-v$ does not. Thus every u-w path in G must include v.

($\Leftarrow$) Now suppose $u, w \in V(G)$ (different from v) are such that every u-w path in G contains v. Then there can be no u-w path in $G - v$, i.e. $G - v$ is disconnected. Therefore v is a cut vertex.

///.

Observe that K_n has no cut vertices. At the other extreme, P_n (for $n \geq 3$) has $n-2$ cut vertices.



P_n

<u>Theorem</u>

Let G be a graph on $n \geq 2$ vertices. Then G has at most $n-2$ cut vertices.

<u>Proof</u>:

If the result holds for connected graphs it clearly holds for disconnected ones, so we assume G is connected.

Assume, to get a contradiction, that G has at least $n-1$ cut vertices. i.e. there is at most one vertex in G which is not a cut vertex.

PICK $u, v \in V(G)$ SUCH THAT

$$d(u,v) = \text{DIAM}(G) := \max\{d(x,y) : x, y \in V(G)\}.$$

SINCE G IS CONNECTED AND $n \geq 2$, WE KNOW $u \neq v$. BY OUR ASSUMPTION, (AT LEAST) ONE OF u AND v IS A CUT VERTEX, SAY v.

THUS $G - v$ IS DISCONNECTED. PICK $w \in V(G)$ LYING IN A COMPONENT OF $G - v$ OTHER THAN THAT OF u. THEN EVERY u-w PATH INCLUDES THE VERTEX v.

IT FOLLOWS THAT A SHORTEST u-w PATH IN G PROPERLY CONTAINS A SHORTEST u-v PATH. THEREFORE $d(u,v) < d(u,w)$, CONTRARY TO OUR CHOICE OF u AND v.

THIS CONTRADICTION SHOWS OUR ASSUMPTION WAS FALSE, AND G HAS AT MOST $n-2$ CUT VERTICES.

///.

LET G BE A CONNECTED GRAPH. A SET $W \subseteq V(G)$ IS CALLED A <u>SEPARATING SET</u> IF $G - W$ IS DISCONNECTED. (RECALL WHEN WE REMOVE A VERTEX, WE REMOVE ALL INCIDENT EDGES.

OBSERVE THAT K_n HAS NO SEPARATING SET. ALSO NOTE THAT IF G IS SIMPLE, AND $G \neq K_n$, THEN G DOES CONTAIN A SEPARATING SET.

<u>EXERCISE</u>: VERIFY THIS.

LET G BE A SIMPLE GRAPH, $G \neq K_n$. WE DEFINE THE <u>(VERTEX) CONNECTIVITY</u>, DENOTED $\kappa(G)$, TO BE THE SIZE OF A SMALLEST SEPARATING SET IN G.

SINCE K_n HAS NO SEPARATING SET WE ADOPT THE CONVENTION THAT $\kappa(K_n) = n-1$, i.e. THE NUMBER OF VERTICES WHOSE REMOVAL WOULD RESULT IN K_1.

THE ANALOGOUS DEFINITIONS CAN BE MADE FOR EDGES. A SET $F \subseteq E(G)$ IS CALLED A <u>DISCONNECTING SET</u> IF $G - F$ IS DISCONNECTED.

NOTE THAT A DISCONNECTING SET WITH JUST ONE EDGE IS A BRIDGE, WHILE A SEPARATING SET WITH ONE VERTEX IS A CUT VERTEX.

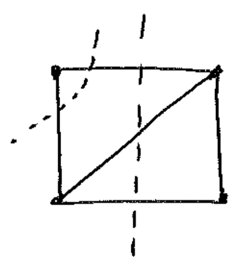
LET G BE A CONNECTED GRAPH WITH AT LEAST TWO VERTICES AND ONE EDGE. THEN G HAS A DISCONNECTING SET.

WE DEFINE THE <u>(EDGE) CONNECTIVITY</u> OF G, DENOTED $\lambda(G)$, TO BE THE SIZE OF THE SMALLEST DISCONNECTING SET. BY CONVENTION $\lambda(K_1) = 0$.

A DISCONNECTING SET $F \subseteq E(G)$ WHICH CONTAINS NO PROPER DISCONNECTING SUBSET IS CALLED A <u>CUTSET</u>.

NOT ALL CUTSETS HAVE THE SAME SIZE.

<u>EX.</u>



$$\lambda(G) = 2$$
$$\kappa(G) = 2$$

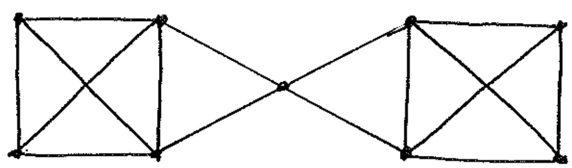
$\lambda(G)$ COULD ALSO BE DEFINED AS THE SIZE OF THE SMALLEST CUTSET.

THEOREM (WHITNEY'S INEQUALITY)
LET G BE A GRAPH WITH AT LEAST TWO VERTICES AND AT LEAST ONE EDGE. DENOTE BY $\delta(G)$ THE MINIMUM VERTEX DEGREE IN G. THEN

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

EXERCISE: PROVE THIS.

EX



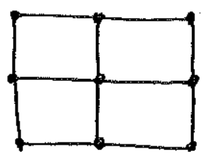
$$\kappa(G) = 1, \quad \lambda(G) = 2, \quad \delta(G) = 3$$

LET $u, v \in V(G)$. A PAIR OF u-v PATHS P AND Q ARE CALLED INTERNALLY DISJOINT IF THE ONLY VERTICES COMMON TO BOTH P AND Q ARE u AND v.

WE CALL G K-CONNECTED IF $\kappa(G) \geq k$. G IS CALLED K-EDGE CONNECTED IF $\lambda(G) \geq k$.

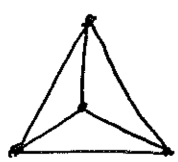
THEOREM (WHITNEY)
LET G BE A SIMPLE GRAPH WITH AT LEAST $K+1$ VERTICES. THEN G IS K-CONNECTED IF AND ONLY IF FOR EACH PAIR OF DISTINCT VERTICES u, v, THERE EXISTS A SET OF K u-v PATHS WHICH ARE PAIRWISE INTERNALLY DISJOINT.

EX.



$\kappa(G) = 2$

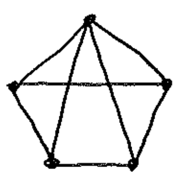
EX.



$\kappa(K_4) = 3$

EX $\kappa(K_n) = n-1$.

EX.



$\kappa(G) = 3$

NOTE G IS CONNECTED IF AND ONLY IF IT IS 1-CONNECTED. A GRAPH WHICH IS 2-CONNECTED IS SOMETIMES CALLED BI-CONNECTED

(PROOF LATER)