2.5 Shortest Path Problems.

Problem: Given a vertex $s \in V(G)$, find a shortest s-v path for any $v \in V(G)$ (if it exists)

The Breadth First Search Algorithm assigns a label $\lambda(v)$ to each vertex v such that $\lambda(v) = d(s,v)$. These labels can then be used to find a shortest path from s to v.

BFS

1.) $\lambda(s) \leftarrow i \leftarrow 0$

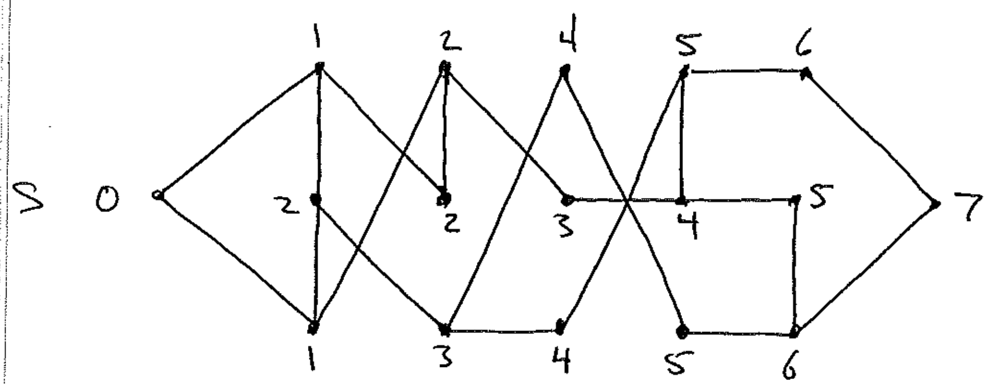
2.) while there are vertices labeled i with unlabeled neighbors.

3.) label all such neighbors $i+1$

4.) $i \leftarrow i+1$

5.) label all remaining vertices ∞.

Ex.



THEOREM.

WHEN BFS IS COMPLETE, $\lambda(v) = d(s,v)$ FOR ALL $v \in V(G)$.

PROOF:

IF $\lambda(v) = \infty$, THEN ALL NEIGHBORS OF v MUST ALSO BE ∞, OTHERWISE v WOULD HAVE BEEN ASSIGNED A FINITE LABEL ON SOME APPLICATION OF (3). THUS ALL VERTICES IN $C(v)$ ARE ∞. BUT $\lambda(s) = 0$, SO $s \notin C(v)$. THEREFORE $d(v,s) = \infty = \lambda(v)$.

NOW SUPPOSE $\lambda(v) < \infty$. WE USE INDUCTION ON $i = \lambda(v)$. i.e. WE SHOW FOR ALL $i = 0, 1, 2, \ldots$ THAT $\lambda(v) = i$ IFF $d(s,v) = i$.

OUR BASE CASE IS SIMPLY $\lambda(s) = 0 = d(s,s)$.

LET $k > 0$ AND ASSUME THE ABOVE CLAIM IS TRUE FOR ALL $i : 1 \leq i \leq k$. WE WILL SHOW $\lambda(v) = k+1$ IFF $d(s,v) = k+1$.

($\Longrightarrow$)

LET $\lambda(v) = k+1$. THEN v WAS LABELED ON THE k^{th} ITERATION OF LOOP 2-4 WHEN IT WAS THE UNLABELED NEIGHBOR OF SOME u WITH $\lambda(u) = k$.

BY OUR INDUCTION HYPOTHESIS $d(s,u)=k$. APPEND EDGE uv TO A SHORTEST s-u PATH TO OBTAIN AN s-v PATH OF LENGTH $k+1$. THUS $d(s,v) \le k+1$.

BUT ALSO $d(s,v) \ge k+1$ SINCE $d(s,v) < k+1$ IMPLIES, BY OUR INDUCTION HYPOTHESIS, $\lambda(v) = d(s,v) \le k$, CONTRADICTING $\lambda(v) = k+1$.

THEREFORE $\lambda(v) = k+1 = d(s,v)$.

$(\Leftarrow)$

SUPPOSE $d(s,v) = k+1$. THEN THERE EXISTS A SHORTEST s-v PATH P WITH $len(P) = k+1$. THE NEXT TO LAST VERTEX, CALL IT w, ON P MUST THEN HAVE $d(s,w) = k$. (RECALL: ANY SEGMENT OF A SHORTEST PATH IS ALSO A SHORTEST PATH.)

BY THE INDUCTION HYPOTHESIS, $\lambda(w) = k$. NOW v MUST BE UNLABELED JUST BEFORE THE k^{TH} ITERATION OF LOOP 2-4, SINCE OTHERWISE $d(s,v) < k+1$, AGAIN BY THE INDUCTION HYPOTHESIS. SINCE v IS ADJACENT TO w, v IS ASSIGNED $\lambda(v) = k+1$ ON THIS k^{TH} ITERATION. $\therefore \lambda(v) = k+1$.

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LET $v \in V(G)$, $k = \lambda(v)$ AFTER BFS.
TO FIND AN s-v PATH OF LENGTH k
DO THE FOLLOWING.

1.) $v_k \leftarrow v$, $i \leftarrow k$
2.) while $i > 0$
3.) CHOOSE u ADJACENT TO v_i WITH $\lambda(u) = i-1$.
4.) $v_{i-1} \leftarrow u$
5.) $i \leftarrow i-1$

THEN $P: s = v_0, v_1, v_2, \ldots, v_k = v$ IS A SHORTEST s-v PATH.

TO FIND THE NUMBER OF SUCH PATHS WE COULD COMPUTE THE sv^{th} ENTRY IN A^k, WHERE $A = (a_{ij})$ IS THE ADJACENCY MATRIX FOR G. i.e. THE NUMBER OF SHORTEST s-v PATHS IS

$$a_{sv}^{(k)} = \sum_{\{i_1, \ldots, i_{n-1}\}} a_{si_1} \cdot a_{i_1 i_2} \cdots a_{i_{n-1} v}$$

$$= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_{n-1}=1}^{n} a_{si_1} \cdot a_{i_1 i_2} \cdots a_{i_{n-1} v}$$

WE CAN COMPUTE THIS FORMULA IN STAGES BY ASSIGNING A NEW LABEL $\mu(u)$ FOR ALL u WITH $\lambda(u) \leq k$ AS FOLLOWS:

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μ(v) ← 1
FOR ALL u ≠ v WITH λ(u) = K
        μ(u) ← 0
i ← K
while i > 0
        FOR ALL u WITH λ(u) = i-1
                μ(u) ← Σ_{w : λ(w) = i} a_uw · μ(w)
        i ← i-1
END while.
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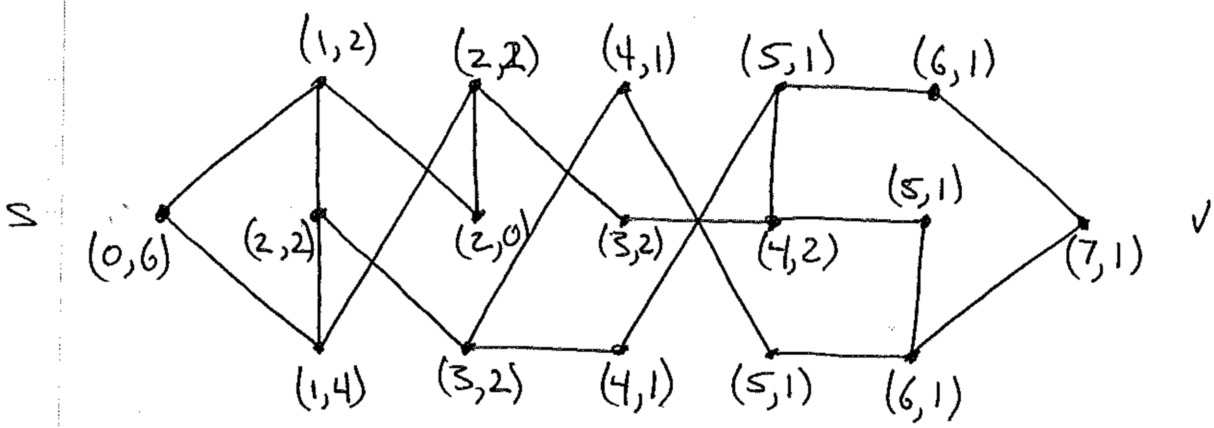
WHEN THIS ALGORITHM IS COMPLETE, $\mu(s)$ IS THE NUMBER OF s-v PATHS OF LENGTH $K = d(s,v)$.

<u>EX.</u> LABELS (λ, μ)



$\therefore$ $\mu(s) = 6$, AND THERE ARE 6 s-v PATHS OF LENGTH 7.

NOW LET $G=(V,E)$ BE A WEIGHTED GRAPH WITH NON-NEGATIVE WEIGHT FUNCTION $w: E \to \mathbb{R}$.

WE MAY THINK OF THE WEIGHT $w(e) \geq 0$ AS THE DISTANCE BETWEEN THE ENDS u,v OF $e \in E$.

FIX $s \in V$. FOR ANY $v \in V$ DEFINE

$$\delta(v) = \min\{w(P) : P \text{ is an } s\text{-}v \text{ path}\}.$$

IF $v \in C(s)$, AND $\delta(v) = \infty$ OTHERWISE.

DIJKSTRA'S ALGORITHM ASSIGNS A LABEL $\lambda(v)$ TO ALL $v \in V$ (POSSIBLY INFINITE) SUCH THAT, BY COMPLETION, $\lambda(v) = \delta(v)$.

<u>DIJKSTRA</u>

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λ(s) ← 0
FOR ALL v ≠ s
      λ(v) ← ∞
T ← V
while T ≠ ∅
      SELECT u ∈ T SUCH THAT λ(u) is MINIMUM.
      FOR ALL v ∈ T ADJACENT TO u
            λ(v) ← min(λ(v), λ(u) + w(uv))
      T ← T − {u}.
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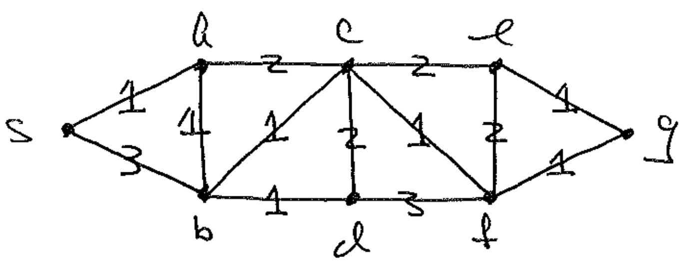
THE SET T CONSISTS OF ALL VERTICES WHOSE LABEL MAY STILL CHANGE, i.e. $V-T$ is THE SET OF "COMPLETED" VERTICES.

BEGIN BY SETTING $\lambda(s)=0$, $\lambda(v)=\infty$ FOR ALL $v \neq s$. FOR EACH v ADJACENT TO s SET $\lambda(v) = w(sv)$, AND MARK s AS COMPLETE BY REMOVING IT FROM T. AMONG ALL VERTICES NOT COMPLETE, PICK u SUCH THAT $\lambda(u)$ is MINIMUM. THEN CHECK IF EACH OF u's NEIGHBORS MINIMUM DISTANCE ESTIMATE $\lambda(v)$ CAN BE IMPROVED BY FIRST PASSING THROUGH u.

EX



	s	a	b	c	d	e	f	g	T
λ	0	∞	∞	∞	∞	∞	∞	∞	$\{\underline{s}, a, b, c, d, e, f, g\}$
	0	1	3	∞	∞	∞	∞	∞	$\{\underline{a}, b, c, d, e, f, g\}$
	0	1	2	3	∞	∞	∞	∞	$\{\underline{b}, c, d, e, f, g\}$
	0	1	2	3	3	∞	∞	∞	$\{c, \underline{d}, e, f, g\}$
	0	1	2	3	3	∞	6	∞	$\{\underline{c}, e, f, g\}$
	0	1	2	3	3	5	4	∞	$\{e, \underline{f}, g\}$
	0	1	2	3	3	5	4	5	$\{e, \underline{g}\}$
	0	1	2	3	3	5	4	5	$\{\underline{e}\}$
	0	1	2	3	3	5	4	5	$\emptyset$

THEOREM

WHEN DIJKSTRA IS COMPLETE, $\lambda(v) = \delta(v)$ FOR ALL $v \in V$.

REMARKS

- ONCE THE LABELS $\lambda(v)$ ARE SET IN (1) AND (3), THEY MAY DECREASE, BUT THEY NEVER INCREASE.
- AFTER u IS DELETED FROM T, $\lambda(u)$ NEVER CHANGES.

LEMMA

IF AFTER DIJKSTRA IS COMPLETE $\lambda(v) = \infty$, THEN $\delta(v) = \infty$.

THE PROOF OF THIS FACT IS SIMILAR TO THE ANALOGOUS RESULT FOR BFS AND WE LEAVE IT AS AN EXCERCISE. (SHOW $\lambda(v) = \infty \Rightarrow v \notin C(s)$.)

LEMMA

IF AT SOME POINT IN DIJKSTRA $\lambda(v) < \infty$, THERE EXISTS AN s-v PATH OF WEIGHT $\lambda(v)$.

PROOF.

WE USE INDUCTION ON THE TIME (i.e. THE PARTICULAR STEP) AT WHICH $\lambda(v)$ IS ASSIGNED A FINITE VALUE.

ON THE FIRST STEP $\lambda(s) = 0$ AND THE TRIVIAL PATH FROM s TO s HAS WEIGHT 0, WHICH COMPLETES THE BASE CASE.

ASSUME $v \neq s$ AND THAT $\lambda(v)$ BECOMES FINITE AT SOME TIME DURING THE EXECUTION OF DIJKSTRA.

ASSUME THAT FOR EVERY FINITE LABEL $\lambda(u)$ ASSIGNED BEFORE $\lambda(v)$, THERE IS AN s-u PATH OF LENGTH $\lambda(u)$.

AT THE TIME $\lambda(v)$ BECOMES FINITE (OR AT ANY OTHER TIME $\lambda(v)$ STEPS DOWN IN VALUE) WE SET IN (8)

$$\lambda(v) = \lambda(u) + w(uv)$$

WHERE u IS ADJACENT TO v, AND HAS BEEN ASSIGNED A (FINITE) LABEL $\lambda(u)$ PRIOR TO v.

BY OUR INDUCTION HYPOTHESIS G CONTAINS AN S-U PATH OF WEIGHT $\lambda(u)$. APPEND TO THIS PATH THE EDGE $e = uv$ TO OBTAIN AN S-V PATH OF WEIGHT $\lambda(v) = \lambda(u) + w(e)$.

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LEMMA

AT THE TIME u IS CHOSEN IN (6), $\lambda(u) = \delta(u)$.

REMARK

IN PARTICULAR, SINCE $\lambda(u)$ NEVER CHANGES AFTER u IS REMOVED FROM T, WE HAVE FOR ALL $v \in V$, $\lambda(v) = \delta(v)$ UPON TERMINATION OF THE ALGORITHM. THIS PROVES THE VALIDITY OF DIJKSTRA.

PROOF:

WE USE INDUCTION ON THE ORDER IN WHICH VERTICES ARE REMOVED FROM T.

THE FIRST VERTEX REMOVED IS $u = s$. IN THIS CASE $\lambda(u) = 0 = \delta(s)$, AND THE BASE OF THE INDUCTION IS VERIFIED.

LET $u \neq s$. ASSUME THAT FOR ALL VERTICES v WHICH ARE REMOVED FROM T BEFORE u, $\lambda(v) = \delta(v)$.

BY THE LAST LEMMA $\lambda(u)$ IS THE WEIGHT OF SOME s-u PATH, WHENCE $\lambda(u) \geq \delta(u)$. WE WILL SHOW THAT $\lambda(u) > \delta(u)$ IS IMPOSSIBLE

LET P BE A MINIMUM WEIGHT PATH FROM s TO u. SAY

$$P: s = v_0, v_1, v_2, \ldots, v_k = u$$

AND SET $e_i = v_{i-1}v_i$ $(1 \leq i \leq k)$. THEN

$$\delta(u) = w(P) = \sum_{i=1}^{k} w(e_i)$$

(ALSO NOTE THAT $\delta(v_{i+1}) = \delta(v_i) + w(e_{i+1})$ FOR ALL i, $1 \leq i \leq k$, SINCE ANY SEGMENT OF A "SHORTEST" PATH IS ALSO A "SHORTEST" PATH.)

LET v_j BE THE LAST VERTEX ON P WHICH WAS REMOVED FROM T BEFORE u WAS.

i.e. ALL VERTICES ON P AFTER v_j WERE REMOVED FROM T AFTER u WAS. (v_{j+1} IN PARTICULAR)

BY OUR INDUCTION HYPOTHESIS

$$\lambda(v_j) = \delta(v_j) = \sum_{i=1}^{j} w(e_i)$$

JUST BEFORE v_j WAS REMOVED FROM T WE HAVE $v_{j+1} \in T$, SO BY (8)

$$\lambda(v_{j+1}) \leq \lambda(v_j) + w(e_{j+1})$$

$$= \delta(v_j) + w(e_{j+1})$$

$$= \delta(v_{j+1})$$

$$\leq \delta(u)$$

THIS LAST INEQUALITY FOLLOWS FROM THE FACT THAT ALL WEIGHTS ARE NON-NEGATIVE: $w(e) \geq 0$.

SINCE λ VALUES DO NOT INCREASE, THE INEQUALITY $\lambda(v_{j+1}) \leq \delta(u)$ STILL HOLDS JUST BEFORE u IS REMOVED FROM T.

IF AT THAT TIME $\delta(u) < \lambda(u)$, THEN

$$\lambda(v_{j+1}) < \lambda(u).$$

NOW IF $u = v_{j+1}$ (i.e. $j+1 = k$) THIS IS A CONTRADICTION, SO SUPPOSE $u \neq v_{j+1}$.

BUT THIS IS ALSO IMPOSSIBLE SINCE $\lambda(v_{j+1}) < \lambda(u)$ VIOLATES THE MINIMALITY ASSUMPTION ON u. (RECALL $v_{j+1} \in T$ AT THIS POINT, AND SEE LINE (6) OF ALGORITHM.)

THEREFORE $\delta(u) < \lambda(u)$ IS IMPOSSIBLE, AND $\delta(u) = \lambda(u)$ AS REQUIRED.

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