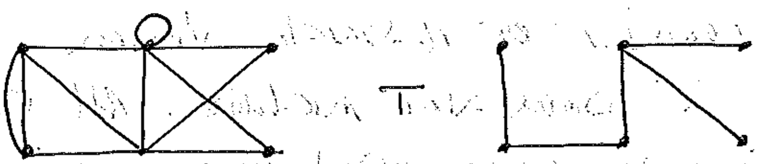
2.3 SPANNING TREES.

DEFN:
A SPANNING TREE IN A GRAPH G IS A SPANNING SUBGRAPH WHICH IS ALSO A TREE.

EX.

G 

THEOREM:
G CONTAINS A SPANNING TREE IF AND ONLY IF G IS CONNECTED.

PROOF:
($\Rightarrow$) SUPPOSE G CONTAINS A SPANNING TREE. THEN ANY TWO VERTICES ARE CONNECTED BY A (UNIQUE) PATH CONSISTING OF TREE EDGES. $\therefore$ G IS CONNECTED.

($\Leftarrow$) SUPPOSE G IS CONNECTED. A SPANNING TREE CAN BE CONSTRUCTED BY THE FOLLOWING ALGORITHM:

1.) while there remains a cycle in G
2.) select such a cycle and remove one of its edges.

On each execution of (2) the resulting graph is still connected since every cycle edge is a non-bridge. The algorithm ends when we reach a connected graph with no cycles, i.e. a tree. This tree spans G since no vertices were removed.

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How many spanning trees are contained in a graph?

Let G be a simple connected graph and define

$$S(G) = \{\text{spanning trees in } G\}.$$

We will develop a formula for $|S(G)|$.

We begin by considering complete graphs $G = K_n$ for various n.

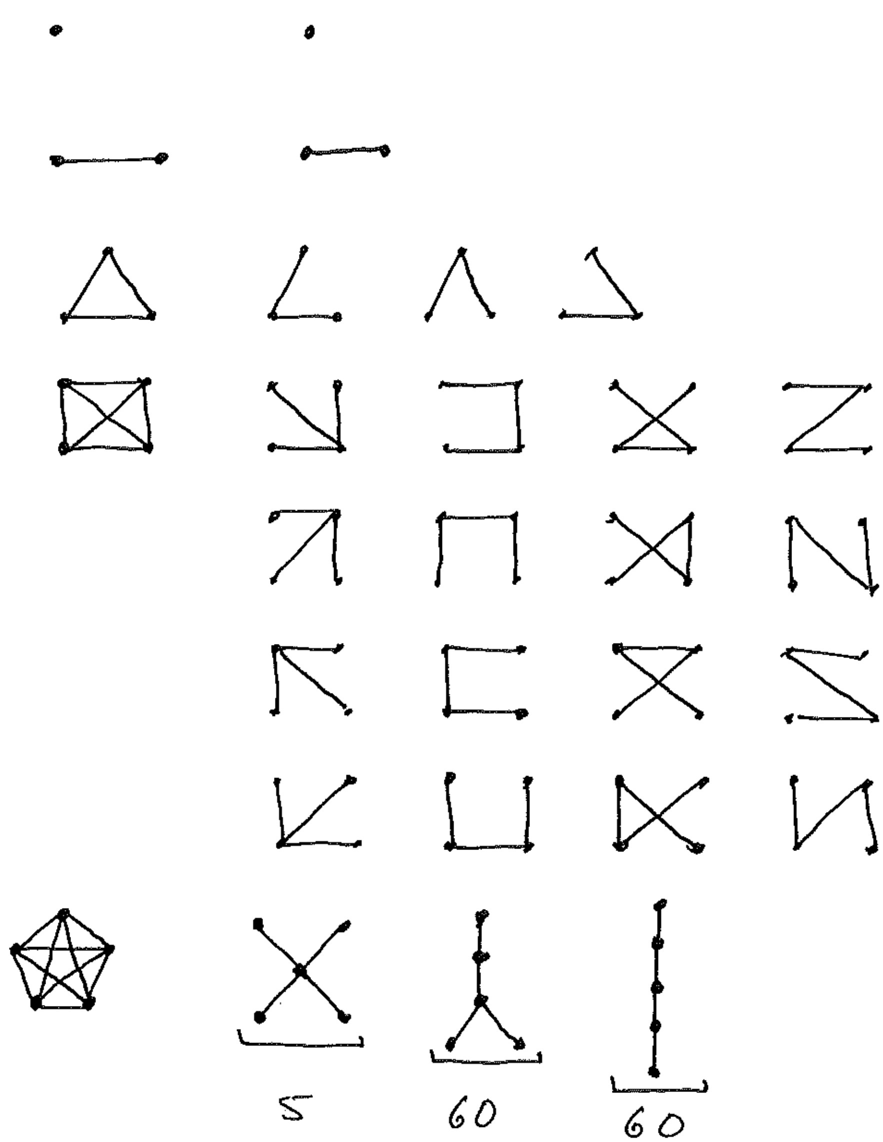
K_1 $\quad 1 = 1^{1-2}$

K_2 $\quad 1 = 2^{2-2}$

K_3 $\quad 3 = 3^{3-2}$

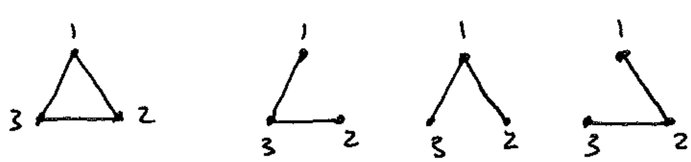
K_4 $\quad 16 = 4^{4-2}$

K_5 $\quad 125 = 5^{5-2}$

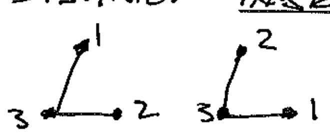


NOTE THE NUMBER OF SPANNING TREES IN K_n EQUALS THE NUMBER OF LABELED TREES ON n VERTICES.

e.g.



THESE ARE DISTINCT LABELED TREES. OBSERVE



ARE IDENTICAL LABELED TREES.

THEOREM (CAYLEY'S THEOREM)

$$|S(K_n)| = n^{n-2}$$

THERE ARE MANY INTERESTING AND BEAUTIFUL PROOFS OF THIS FACT.

WE WILL GO FURTHER AND FIND A FORMULA FOR $|S(G)|$ FOR ANY SIMPLE CONNECTED GRAPH G, AND CAYLEY'S THEOREM WILL FOLLOW AS A SPECIAL CASE.

LET $V(G)=\{v_1,\dots,v_n\}$, $E(G)=\{e_1,\dots,e_m\}$ AND $M = M(G)$ BE THE INCIDENCE MATRIX.

SINCE G IS SIMPLE M IS A 0-1 MATRIX WITH EXACTLY TWO 1'S IN EACH COLUMN. ARBITRARILY REPLACE ONE 1 BY -1 IN EACH COLUMN.

$$\vec{M} = \begin{array}{c} k \\ \left(\begin{array}{c} \\ -1 \\ 1 \\ \end{array}\right) \end{array} \begin{array}{c} \\ i \\ j \end{array} \qquad \overset{-1}{v_i} \xrightarrow{e_k} \overset{+1}{v_j}$$

ORIENTED INCIDENCE MATRIX

THIS AMOUNTS TO CHOOSING AN ORIENTATION ON EACH EDGE. DENOTE THE RESULTING MATRIX BY $\vec{M}$.

NOTE: ROWS OF $\vec{M}$ ADD TO THE ZERO VECTOR.

DEFN

THE LAPLACIAN OF G IS THE $n \times n$ SYMMETRIC MATRIX GIVEN BY

$$L = L(G) = \vec{M}\,\vec{M}^T$$

LET L_{rr} DENOTE THE r-r COFACTOR OF L, i.e. THE MATRIX OBTAINED BY DELETING THE r^{TH} ROW AND r^{TH} COLUMN FROM L.

MATRIX TREE THEOREM (KIRCHOFF)

FOR ANY r, $1 \leq r \leq n$

$$|S(G)| = \det(L_{rr}).$$

BEFORE WE PROVE THIS CELEBRATED THEOREM, WE NOTE THERE IS AN ALTERNATE WAY TO DEFINE THE LAPLACIAN L.

LET $D = D(G)$ BE THE DEGREE MATRIX OF G, i.e. THE MATRIX WITH THE VERTEX DEGREES OF G ALONG THE MAIN DIAGONAL, AND ZEROS ELSEWHERE. THEN

$$L = D - A$$

WHERE $A = A(G)$ IS THE ADJACENCY MATRIX.

EXCERCISE

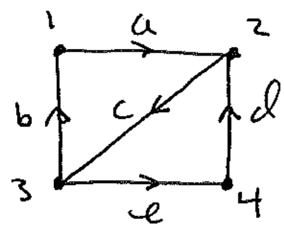
SHOW THESE DEFINITIONS ARE EQUIVALENT, I.E. SHOW

$$\vec{M}\vec{M}^T = D - A$$

OBSERVE THE RIGHT HAND SIDE IS INDEPENDENT OF EDGE LABELS AND ORIENTATIONS AND SO MUST BE THE LEFT SIDE, ALTHOUGH THIS IS NOT OBVIOUS.

EX

G



$$\vec{M} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \\ \begin{matrix} a & \;b & \;c & \;d & \;e \end{matrix} \end{array}$$

$$L = \vec{M}\vec{M}^T = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

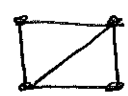
OBSERVE: $L = D - A$.

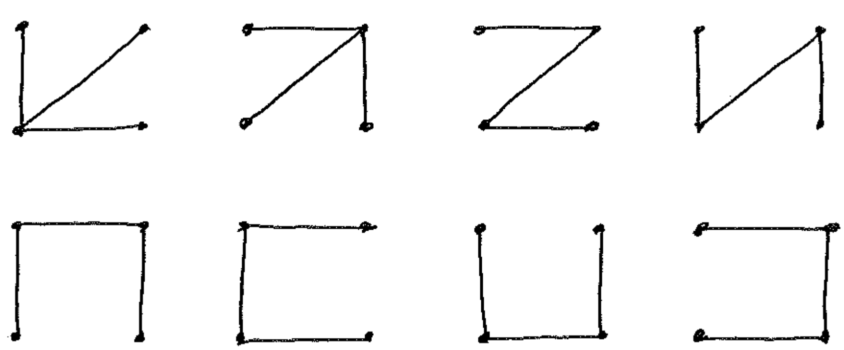
NOW PICK $r = 2$ IN THE MATRIX TREE THEOREM. THEN

$$L_{22} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

AND

$$\det(L_2) = 2(6-1) - (-1)(-2-0) + 0 = 8$$

THE 8 SPANNING TREES OF  ARE



TO PROVE THE MATRIX TREE THEOREM WE USE SOME FACTS FROM LINEAR ALGEBRA.

LET P BE AN $s \times t$ MATRIX AND Q A $t \times s$ MATRIX WHERE $s \le t$. THEN PQ IS AN $s \times s$ MATRIX WHOSE DETERMINANT IS GIVEN BY THE CAUCHY-BINET THEOREM.

LET $I = \{ i_1, i_2, \dots, i_s \}$ BE A SET OF INDICES WITH $1 \le i_1 < i_2 < \dots < i_s \le t$, AND LET

P_I = $s \times s$ SUBMATRIX WITH COLUMNS INDEXED BY I

Q^I = $s \times s$ SUBMATRIX WITH ROWS INDEXED BY I.

<u>THEOREM</u> (CAUCHY-BINET)

$$\det PQ = \sum_I \det P_I \cdot \det Q^I .$$

<u>PROOF OF MTT</u>:
LET $\vec{M}_r$ BE THE $(n-1) \times m$ MATRIX OBTAINED BY DELETING THE r^{th} ROW FROM $\vec{M}$. THEN

$$L_{rr} = \vec{M}_r \vec{M}_r^T$$

AND BY CAUCHY-BINET: $(s = n-1 \le m = t)$

ASSUME G CONN.

$$\det L_{rr} = \sum_{N} \det N \cdot \det N^{T} = \sum_{N} (\det N)^{2}$$

WHERE N RUNS THROUGH ALL $(n-1) \times (n-1)$ SUBMATRICES OF $\vec{M}_r$.

EACH SUCH MATRIX N CORRESPONDS TO A CHOICE OF $(n-1)$ COLUMNS IN $\vec{M}_r$, AND HENCE IN $\vec{M}$, AND HENCE TO A SET OF $(n-1)$ EDGES IN $E(G)$.

WE WILL SHOW THAT $\det N = \pm 1$ IF THOSE EDGES FORM A SPANNING TREE IN G, AND $\det N = 0$ OTHERWISE. THE RESULT FOLLOWS.

FIRST SUPPOSE THE $(n-1)$ EDGES CORRESPONDING TO N DO NOT FORM A TREE. THE SPANNING SUBGRAPH ON THOSE EDGES MUST THEN BE DISCONNECTED. THIS SUBGRAPH MUST HAVE A COMPONENT WHICH DOES NOT CONTAIN V_r.

THE ROWS OF N CORRESPONDING TO THE VERTICES IN THIS COMPONENT CONSTITUTE THE (ORIENTED) INCIDENCE MATRIX FOR THAT COMPONENT.

THESE ROWS OF N, CONSIDERED AS VECTORS, THEREFORE ADD TO THE ZERO VECTOR. THUS N IS SINGULAR, AND $\det N = 0$.

NOW SUPPOSE THE SPANNING SUBGRAPH CORRESPONDING TO N IS A TREE. WE CHOOSE INDICES $i_1, \dots, i_{n-1}$ AND $j_1, \dots, j_{n-1}$ AS FOLLOWS.

PICK A LEAF $v_{i_1} \neq v_r$ IN THIS TREE WITH e_{j_1} ITS (SOLE) INCIDENT EDGE. DELETE v_{i_1} AND e_{j_1} TO GET A NEW TREE. AGAIN THERE IS A LEAF $v_{i_2} \neq v_r$ AND INCIDENT EDGE e_{j_2} WHICH WE DELETE.

CONTINUE IN THIS WAY UNTIL VERTICES $v_{i_1}, \dots, v_{i_{n-1}}$ AND EDGES $e_{j_1}, \dots, e_{j_{n-1}}$ HAVE BEEN DETERMINED. REARRANGE THE ROWS AND COLUMNS OF N SO THAT v_{i_k} CORRESPONDS TO ROW k, AND e_{j_k} CORRESPONDS TO COLUMN k, FOR $1 \le k \le n-1$.

OBSERVE THAT v_{i_k} IS NOT INCIDENT WITH e_{j_l} FOR $1 \le k < l \le n-1$. HENCE N IS LOWER TRIANGULAR.

ALSO SINCE v_{i_k} IS INCIDENT WITH e_{j_k} THE kk^{TH} ENTRY IN N IS $+1$ OR -1.
THUS

$$\det N = \pm 1$$

AS CLAIMED.

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PROOF OF CAYLEY'S THEOREM.

LET $G = K_n$. THEN

$$L = D - A = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix} \qquad (n \times n)$$

AND

$$L_{rr} = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & & & & \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix} \qquad (n-1) \times (n-1)$$

RECALL THAT BY ADDING A ROW (RESP. COLUMN) TO ANOTHER ROW (RESP. COLUMN) WE DO NOT CHANGE THE VALUE OF THE DETERMINANT.

SUBTRACT COLUMN 1 FROM ALL OTHERS:

$$\begin{pmatrix} n-1 & -n & -n & \cdots & -n \\ -1 & n & 0 & \cdots & 0 \\ -1 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & n \end{pmatrix} \quad (n-1)\times(n-1)$$

ADD EVERY ROW TO ROW 1:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & n & 0 & \cdots & 0 \\ -1 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & n \end{pmatrix} \quad (n-1)\times(n-1)$$

THIS LAST MATRIX HAS DETERMINANT n^{n-2}.

$$\therefore \quad |S(K_n)| = \det L_{rr} = n^{n-2}.$$

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