2.2 BRIDGES

LEMMA

LET $e \in E(G)$. THEN $\omega(G) \le \omega(G-e) \le \omega(G)+1$.

IF $\omega(G-e) = \omega(G)+1$ THE EDGE e IS CALLED A BRIDGE.

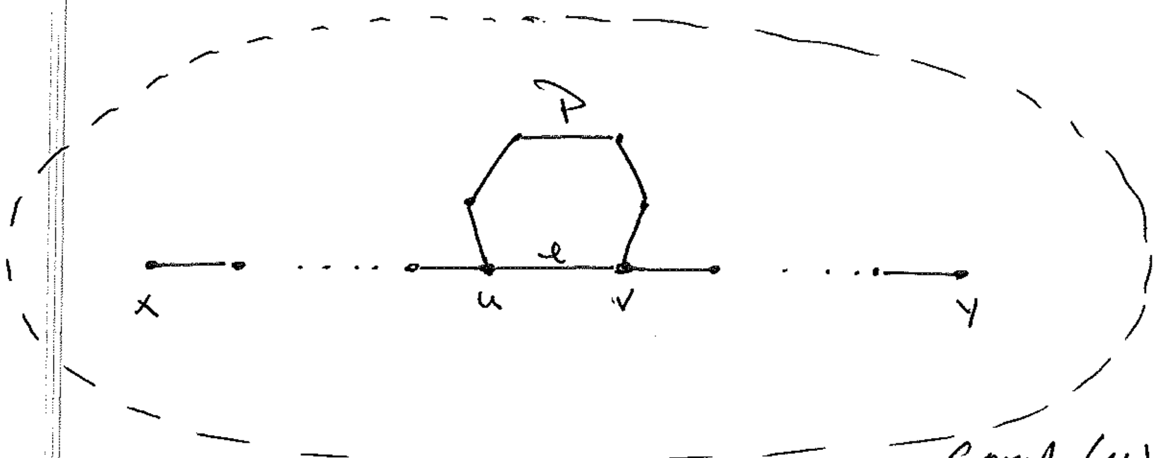
PROOF:

IF e IS A LOOP THEN OBVIOUSLY $\omega(G-e) = \omega(G)$ SO WE ASSUME THIS IS NOT THE CASE. LET u, v BE THE (DISTINCT) ENDS OF e. THERE ARE TWO CASES TO CONSIDER.

CASE 1:

THERE EXISTS A u-v PATH P IN G WHICH DOES NOT INCLUDE e. IN THIS CASE $\omega(G-e) = \omega(G)$ SINCE ANY PATH IN $\mathrm{comp}_G(u)$ WHICH INCLUDES e CAN BE REPLACED BY ONE WHICH DOES NOT BY JUST SPLICING IN P (AND PERHAPS DELETING SOME REDUNDANT EDGES.)

(NOTATION: $\mathrm{Comp}_G(u) = G[C_G(u)]$.)



$Comp_G(u) = Comp_G(v)$.

THUS $C_G(u) = C_{G-e}(u)$, HENCE $Comp_G(u) = Comp_{G-e}(u)$,

AND THEREFORE $\omega(G) = \omega(G-e)$.

CASE 2.

G CONTAINS NO PATH FROM u TO v WHICH DOES NOT CONTAIN e. IN THIS CASE $\omega(G-e) = \omega(G)+1$.

FIRST OBSERVE THAT $C_{G-e}(u) \cap C_{G-e}(v) = \phi$ SINCE ANY u-v PATH MUST CONTAIN e.

ALSO NOTE THAT $C_{G-e}(u) \cup C_{G-e}(v) = C_G(u)$.

TO SEE THIS LET $x \in C_G(u)$ BUT $x \notin C_{G-e}(u)$. THE x-u PATH IN G MUST HAVE e AS ITS LAST EDGE. DELETING THIS EDGE GIVES US AN x-v PATH IN $G-e$, WHENCE $x \in C_{G-e}(v)$.

THUS $Comp_G(u)$ BECOMES, UPON THE REMOVAL OF e, THE TWO COMPONENTS

$Comp_{G-e}(u)$ AND $Comp_{G-e}(v)$. SINCE THE OTHER COMPONENTS ARE UNAFFECTED, THE NUMBER OF COMPONENTS INCREASES BY EXACTLY ONE: $\omega(G-e) = \omega(G) + 1$

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LEMMA.
IF G IS ACYCLIC, THEN EVERY EDGE IS A BRIDGE.

PROOF:
ASSUME (TO GET A CONTRADICTION) THAT $e \in E(G)$ IS A NON-BRIDGE. LET u, v BE ITS (NECESSARILY DISTINCT) ENDS. SINCE $G-e$ IS CONNECTED IT CONTAINS A u-v PATH. ADDING e TO THIS PATH YIELDS A CYCLE IN G, CONTRADICTING THAT G IS ACYCLIC. THEREFORE, NO SUCH EDGE CAN EXIST, i.e. EVERY EDGE IS A BRIDGE.

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NOTE: THE CONVERSE IS ALSO TRUE. PROOF LATER.

LEMMA

LET G BE A GRAPH WITH n VERTICES, m EDGES AND K COMPONENTS. THEN

$$m \geq n-K$$

PROOF:

WE USE INDUCTION ON m. THE BASE CASE $m=0$ IS TRIVIAL SINCE THEN $n=K$, AND THE INEQUALITY REDUCES TO $0 \leq 0$.

LET $m > 0$ AND ASSUME THE RESULT IS TRUE OF GRAPHS WITH FEWER THAN m EDGES. LET $e \in E(G)$ AND FORM THE SUBGRAPH $G-e$ WITH n VERTICES AND $m-1$ EDGES. THERE ARE TWO CASES.

CASE 1: $\omega(G-e) = \omega(G) = K$.

BY THE INDUCTION HYPOTHESIS APPLIED TO $G-e$ $m-1 \geq n-K$, SO THAT $m \geq n-K$, AS REQUIRED.

CASE 2: $\omega(G-e) = \omega(G)+1 = K+1$.

THE INDUCTION HYPOTHESIS APPLIED TO $G-e$ NOW YIELDS

$$m-1 \geq n-(K+1)$$

WHENCE AGAIN $m \geq n-K$.

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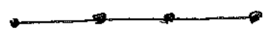
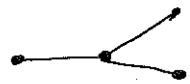
TREES CAN BE CHARACTERIZED IN SEVERAL WAYS

THEOREM. (TREENESS)
LET T BE A GRAPH ON n VERTICES. THE FOLLOWING CONDITIONS ARE EQUIVALENT.

(1) T is A TREE
(2) T is ACYCLIC WITH $m=n-1$ EDGES.
(3) T is CONNECTED WITH $m=n-1$ EDGES.
(4) T is CONNECTED AND EACH EDGE is A BRIDGE.
(5) ANY TWO VERTICES OF T ARE CONNECTED BY A UNIQUE PATH.
(6) T is ACYCLIC, BUT THE ADDITION OF AN EDGE CREATES A UNIQUE CYCLE.

EX. $n=2$

EX. $n=3$ 

EX. $n=4$  

PROOF:

THE SIX CONDITIONS ARE OBVIOUSLY EQUIVALENT IF $n=1$, SO ASSUME $n \geq 2$.

$(1) \Rightarrow (2)$ (INDUCTION)

LET T BE A TREE ON n VERTICIES. ANY OF THE PRECEEDING EXAMPLES CAN BE TAKEN AS OUR BASIS STEP. ASSUME THE THEOREM IS TRUE FOR ANY TREE ON FEWER THAN n VERTICIES, SINCE T CONTAINS NO CYCLES, THE REMOVAL OF ANY EDGE DISCONNECTS T INTO TWO GRAPHS, EACH OF WHICH IS A TREE WITH FEWER THAN n VERTICIES, SAY n_1 AND n_2 VERTICIES RESPECTIVELY. REPLACING THE EDGE THEN GIVES A GRAPH WITH $(n_1-1)+(n_2-1)+1 = (n_1+n_2)-1 = n-1$ EDGES, AS REQUIRED. //

By lemma

By ind. hyp.

$(2) \Rightarrow (3)$

LET T BE A GRAPH WITH NO CYCLES AND $n-1$ EDGES. SUPPOSE T HAS K COMPONENTS, EACH WITH n_i VERTICIES $(1 \leq i \leq K)$. THEN THE

i^{th} COMPONENT IS A CONNECTED GRAPH WITH NO CYCLES, AND HENCE HAS $n_i - 1$ EDGES (BY PREVIOUS STEP). THE TOTAL NUMBER OF EDGES IN T IS

$$n-1 = \sum_{i=1}^{k} (n_i - 1) = n - k,$$

FROM WHENCE IT FOLLOWS $k=1$, AND T IS CONNECTED. //

(3) ⇒ (4)

LET T BE A CONNECTED GRAPH WITH $n-1$ EDGES. RECALL THAT A SIMPLE GRAPH ON n VERTICES, m EDGES, AND k COMPONENTS SATISFIES

$$n - k \leq m$$

LET e BE ANY EDGE OF T. THEN $T - e$ HAS n VERTICES, $n-2$ EDGES AND k COMPONENTS WHERE $n-k \leq n-2$; THUS $k \geq 2$, SHOWING THAT $T-e$ IS DISCONNECTED, i.e. e IS A BRIDGE.

(4) ⇒ (5)

SUPPOSE T IS A CONNECTED GRAPH, EACH OF WHOSE EDGES IS A BRIDGE.

SINCE T IS CONNECTED EACH PAIR OF VERTICIES IS CONNECTED BY AT LEAST ONE PATH. IF THERE WERE A PAIR OF VERTICIES CONNECTED BY TWO DISTINCT PATHS, THEN REMOVING AN EDGE FROM EITHER PATH RESULTS IN A GRAPH WHICH IS STILL CONNECTED, CONTRADICTING THAT EACH EDGE IS A BRIDGE. HENCE ANY PAIR OF VERTICIES IS CONNECTED BY EXACTLY ONE PATH. //

(5) $\Rightarrow$ (6)

SUPPOSE EACH PAIR OF VERTICIES OF T ARE CONNECTED BY A UNIQUE PATH. IF T CONTAINED A CYCLE, THEN ANY PAIR OF VERTICIES ON THAT CYCLE WOULD BE CONNECTED BY TWO DISTINCT PATHS, A CONTRADICTION. THUS T CONTAINS NO CYCLES. LET X, Y BE NON-ADJACENT VERTICIES OF T AND P THE UNIQUE PATH CONNECTING THEM. IF WE JOIN X AND Y WITH A NEW EDGE, THEN THAT EDGE, ALONG WITH P FORMS A CYCLE. (IF MORE THAN ONE CYCLE IS FORMED, THEN X, Y MUST HAVE BEEN CONNECTED BY MORE THAN ONE PATH IN T, A CONTRADICTION.) //

(6) $\Rightarrow$ (1)

SUPPOSE T CONTAINS NO CYCLES, BUT THE ADDITION OF AN EDGE CREATES EXACTLY ONE CYCLE. LET x, y BE TWO NON-ADJACENT VERTICIES OF T. JOIN x AND y BY A NEW EDGE e TO CREATE A SINGLE CYCLE C IN $T+e$. THEN $C-e$ IS A PATH IN T FROM x TO y. THEREFORE T IS CONNECTED. ////

COROLLARY

A FOREST WITH n VERTICIES AND k COMPONENTS MUST HAVE $n-k$ EDGES.

PROOF:

THE i^{TH} COMPONENT IS A TREE, WITH SAY n_i VERTICIES $(1 \le i \le k)$. BY (2) OR (3) THIS COMPONENT HAS $n_i - 1$ EDGES. THE TOTAL NUMBER OF EDGES IS THEN

$$\sum_{i=1}^{k} (n_i - 1) = \left(\sum_{i=1}^{k} n_i\right) - k = n - k.$$

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EXCERCISE:

PROVE THE CONVERSE: A GRAPH ON n VERTICIES, k COMPONENTS, AND $n-k$ EDGES IS NECESSARILY ACYCLIC.