1.7 MATRIX REPRESENTATIONS.

DEFN

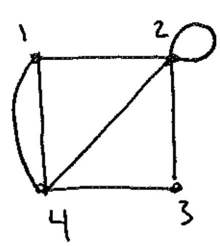
LET G BE A GRAPH ON n VERTICES: $V(G) = \{v_1, v_2, \ldots, v_n\}$. THE ADJACENCY MATRIX OF G IS THE $n \times n$ MATRIX

$$A(G) = (a_{ij})$$

WHERE a_{ij} = NUMBER OF EDGES JOINING v_i TO v_j FOR $1 \le i \le n$, $1 \le j \le n$.

NOTE $A(G)$ IS NECESSARILY SYMMETRIC (i.e. $A(G)^T = A(G)$). THE ADJACENCY MATRIX OF A DIRECTED GRAPH MAY BE NON-SYMMETRIC.

EX.



$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

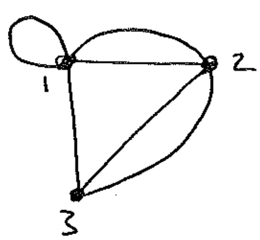
OBSERVE IF G HAS NO LOOPS, THEN THE MAIN DIAGONAL IN $A(G)$ IS ZERO. IF G HAS NO PARALLEL EDGES THEN ALL ENTRIES ARE EITHER ZERO OR ONE.

NOTE THAT $A(G)$ DEPENDS ON THE PARTICULAR LABELING OF THE VERTICES OF G. i.e. IF $G_1 \cong G_2$ THEN WE MAY HAVE $A(G_1) \neq A(G_2)$. HOWEVER $A(G_1)$ AND $A(G_2)$ CAN DIFFER ONLY BY A (SIMULTANEOUS) PERMUTATION OF ROWS AND COLUMNS.

<u>THEOREM</u>

LET G BE A GRAPH ON n VERTICES: $V(G) = \{v_1, v_2, \ldots, v_n\}$ AND $A = A(G)$. THEN FOR ANY $k \geq 0$, THE i-j^{TH} ENTRY OF A^k IS THE NUMBER OF $v_i - v_j$ WALKS IN G OF LENGTH k

<u>EX.</u>



$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix},$$

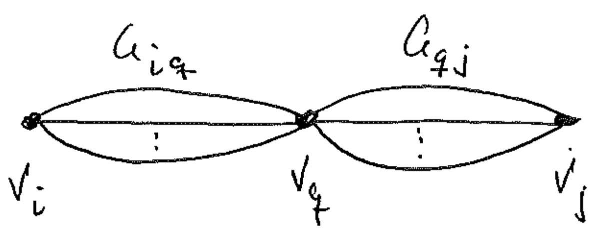
$$A^2 = \begin{pmatrix} 6 & 4 & 5 \\ 4 & 8 & 2 \\ 5 & 2 & 5 \end{pmatrix}$$

Proof:

Let $A = (a_{ij})$, so that a_{ij} is the number of edges in G joining v_i to v_j.

We begin by considering the case $k = 2$. Let $1 \le q \le n$ and observe that

$$a_{iq} a_{qj} = \#\text{ of walks of length 2 from } v_i \text{ to } v_j \text{ which pass through } v_q$$



By summing over all intermediate vertices v_q we get the total number of walks in G from v_i to v_j of length 2.

But $b_{ij} = \sum_{q=1}^{n} a_{iq} a_{qj}$ is the i-j th entry in A^2, i.e.:

$$A^2 = (b_{ij}).$$

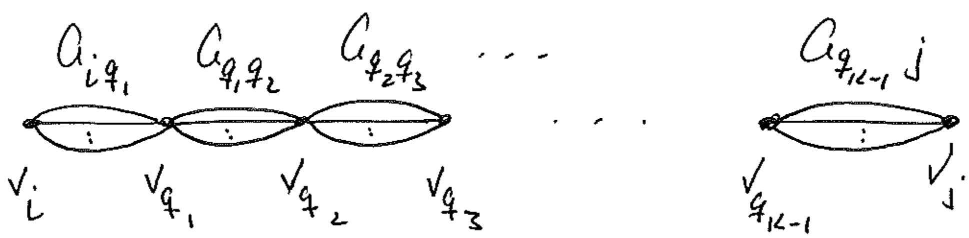
Now to the general case.

Consider $k-1$ vertices $v_{q_1}, v_{q_2}, \ldots, v_{q_{k-1}}$.

OBSERVE THAT

$$a_{iq_1} a_{q_1 q_2} a_{q_2 q_3} \cdots a_{q_{k-1} j}$$

$=$ # OF WALKS OF LENGTH k FROM v_i TO v_j WHICH PASS THROUGH $v_{q_1} \to v_{q_2} \to \cdots \to v_{q_{k-1}}$.



BY SUMMING OVER ALL POSSIBLE INTERMEDIATE VERTICES $q_1, q_2, \ldots, q_{k-1}$, WE OBTAIN THE TOTAL NUMBER OF WALKS OF LENGTH k FROM v_i TO v_j.

BUT

$$c_{ij} = \sum_{q_1} \sum_{q_2} \cdots \sum_{q_{k-1}} a_{iq_1} a_{q_1 q_2} \cdots a_{q_{k-1} j}$$

IS JUST THE ij^{TH} ENTRY OF A^k, I.E.

$$A^k = (c_{ij}) .$$

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THEOREM

LET $V(G) = \{v_1, \ldots, v_n\}$ AND $A = A(G)$.
LET

$$B = A + A^2 + A^3 + \cdots + A^{n-1}.$$

THEN G IS CONNECTED IF AND ONLY IF B IS NON-ZERO OFF THE DIAGONAL, i.e. $B = (b_{ij})$ AND $i \neq j \Rightarrow b_{ij} \neq 0$.

PROOF:
LET $a_{ij}^{(k)}$ DENOTE THE ij^{TH} ENTRY IN A^k, WHICH BY THE PREVIOUS THEOREM IS THE NUMBER OF v_i-v_j WALKS OF LENGTH k. THEN THE ij^{TH} ENTRY IN B IS

$$b_{ij} = a_{ij}^{(1)} + a_{ij}^{(2)} + a_{ij}^{(3)} + \cdots + a_{ij}^{(n-1)}.$$

THUS b_{ij} IS THE NUMBER OF v_i-v_j WALKS OF LENGTH LESS THAN n.

IF G IS CONNECTED THEN EACH PAIR OF DISTINCT VERTICES v_i, v_j $(i \neq j)$ ARE CONNECTED BY A PATH. SINCE G HAS n VERTICES THIS PATH IS OF LENGTH AT MOST $n-1$.

THUS $b_{ij} \neq 0$.

CONVERSELY, IF $b_{ij} \neq 0$ FOR ALL $i \neq j$, THEN EACH PAIR OF DISTINCT VERTICES v_i, v_j ARE JOINED BY A WALK OF LENGTH AT MOST $n-1$. IN PARTICULAR v_i IS CONNECTED TO v_j SO G IS CONNECTED.

REPLACE WALK BY A PATH

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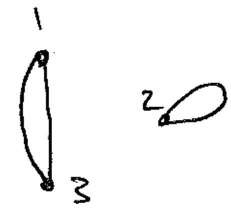
EX.



$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

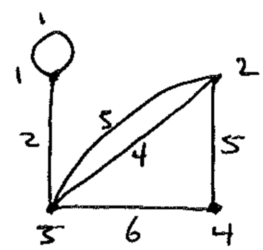
$$B = A + A^2 = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}$$

DEFN:
LET $V(G) = \{v_1, v_2, \ldots, v_n\}$ AND $E(G) = \{e_1, \ldots, e_t\}$. THE <u>INCIDENCE MATRIX</u> $M = M(G)$ OF G IS THE $n \times t$ MATRIX WHOSE ij^{TH} ENTRY IS THE NUMBER OF TIMES VERTEX v_i IS INCIDENT WITH EDGE e_j.

i.e. if $M = (m_{ij})$ then

$$m_{ij} = \begin{cases} 0 & v_i \text{ not incident with } e_j \\ 1 & v_i, e_j \text{ are incident, } e_j \text{ not loop} \\ 2 & v_i, e_j \text{ are incident, } e_j \text{ is loop.} \end{cases}$$

Ex.



$$M(G) = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

NOTE THAT $M(G)$ DEPENDS ON BOTH THE VERTEX LABELS AND EDGE LABELS.

THE SUM OF ANY ROW IN $M(G)$ IS THE DEGREE OF THE CORRESPONDING VERTEX, WHILE THE SUM OF ANY COLUMN IS 2.

THE SUM OF ANY ROW OR COLUMN IN $A(G)$ IS THE DEGREE OF THE CORRESPONDING VERTEX ONLY IF G HAS NO LOOPS.