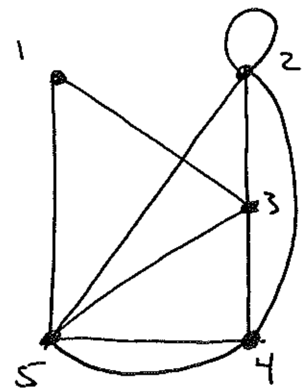
1.4 VERTEX DEGREE

DEFN:
LET $v \in V(G)$. THE <u>DEGREE</u> $d(v)$ OF v IN G IS THE NUMBER OF EDGES IN G WHICH ARE INCIDENT WITH v (COUNTING LOOPS TWICE.)

i.e. $d(v)$ is literally THE NUMBER OF "INCIDENCES" WITH VERTEX v.

EX.



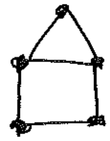
$d(1) = 2$
$d(2) = 5$
$d(3) = 4$
$d(4) = 4$
$d(5) = 5$

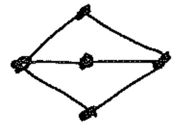
THE <u>DEGREE SEQUENCE</u> OF A GRAPH IS THE SET OF VERTEX DEGREES ARRANGED IN INCREASING ORDER.

e.g. ABOVE: 2, 4, 4, 5, 5.

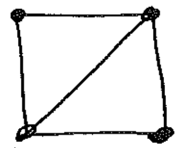
NOTE THAT ISOMORPHIC GRAPHS MUST HAVE THE SAME DEGREE SEQUENCE. THE CONVERSE IS FALSE.

EX. 22233

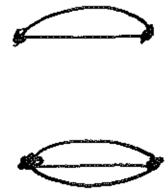




EX.



2, 2, 3, 3



2, 2, 3, 3

THEOREM

LET $V(G) = \{v_1, v_2, \dots, v_n\}$ AND $|E(G)| = e$.
THEN

$$\sum_{i=1}^{n} d(v_i) = 2e$$

THIS IS SOMETIMES CALLED THE <u>HANDSHAKE LEMMA</u>.

PROOF.

EACH EDGE CONTRIBUTES EXACTLY TWO INCIDENCES TO THE ABOVE SUM. ///.

CALL A VERTEX EVEN IF IT HAS EVEN DEGREE, ODD IF IT HAS ODD DEGREE.

COROLLARY

ANY GRAPH HAS AN EVEN NUMBER OF ODD VERTICES.

PROOF.

LET $X \subseteq V$ BE THE SET OF EVEN VERTICES, AND $Y \subseteq V$ THE ODD VERTICES. THEN

$$2e = \sum_{v \in V} d(v) = \sum_{u \in X} d(u) + \sum_{w \in Y} d(w)$$

NOW $\sum_{u \in X} d(u)$ IS A SUM OF EVEN NUMBERS SO MUST BE EVEN. THUS

$$\sum_{w \in Y} d(w) = 2e - \sum_{u \in X} d(u)$$

MUST ALSO BE EVEN. IF $|Y|$ WERE ODD THEN THE LHS WOULD BE A SUM OF AN ODD NUMBER OF ODD NUMBERS, AND HENCE WOULD BE ODD. SINCE THIS IS NOT THE CASE, $|Y|$ MUST BE EVEN AS CLAIMED.

///.

DEFN

A GRAPH G IS CALLED K-REGULAR IF EACH $v \in V(G)$ HAS DEGREE K, i.e. $d(v) = K$.

OBSERVE THAT A K-REGULAR GRAPH ON n VERTICES SATISFIES

$$kn = \sum d(v) = 2e$$

WHENCE $e = \frac{kn}{2}$.

Ex.

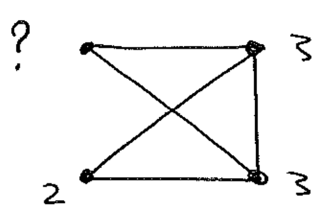
THE COMPLETE GRAPH K_n IS $(n-1)$-REGULAR SINCE EACH VERTEX IS JOINED TO EVERY OTHER VERTEX. THUS

$$|E(K_n)| = \frac{n(n-1)}{2}$$

AS WE ALREADY SAW.

Ex.

SHOW THERE IS NO SIMPLE GRAPH WITH DEGREE SEQUENCE 2 3 3 4.



A SEQUENCE WHICH IS THE DEGREE SEQUENCE OF SOME SIMPLE GRAPH IS CALLED GRAPHICAL