1.3 MORE DEFINITIONS

OBSERVE IN THE LAST EXAMPLE THAT THE COMBINATORIAL INFORMATION CONTAINED IN A GRAPH IS INDEPENDENT OF THE WAY IN WHICH THE GRAPH IS DRAWN.

DEFN.
TWO GRAPHS $G_1 = (V_1, E_1)$ AND $G_2 = (V_2, E_2)$ ARE SAID TO BE ISOMORPHIC IF THERE EXIST BIJECTIONS

$$f : V_1 \to V_2 \quad \text{AND} \quad h : E_1 \to E_2$$

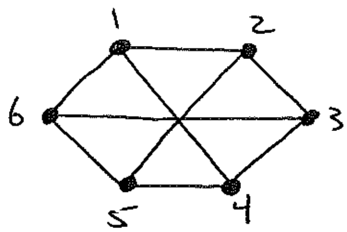
SATISFYING THE FOLLOWING CONDITION :

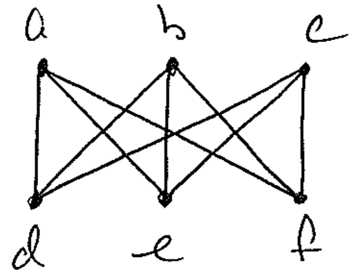
$e \in E_1$ joins $x, y \in V_1$ if and only if $h(e) \in E_2$ joins $f(x), f(y) \in V_2$.

IN OTHER WORDS, THE CORRESPONDENCES f AND h PRESERVE ALL ADJACENCY AND INCIDENCE RELATIONS, i.e. G_1 AND G_2 ARE REALLY THE SAME GRAPH WITH DIFFERENT LABELS.

THE PAIR (f, h) is CALLED A GRAPH ISOMORPHISM.

EX.





ISOMORPHISM:

V: $1 \to a$, $2 \to d$, $3 \to b$, $4 \to e$, $5 \to c$, $6 \to f$

E: $12 \to ad$, $14 \to ae$, $16 \to af$, $23 \to db$,
$25 \to dc$, $34 \to be$, $36 \to bf$, $45 \to ec$, $56 \to cf$

ANOTHER ISOMORPHISM

V: $1 \to b$, $2 \to d$, $3 \to c$, $4 \to f$, $5 \to a$, $6 \to e$

E: ....

EXERCISE

WRITE DOWN A FEW MORE SUCH ISOMORPHISMS. HOW MANY ISOMORPHISMS ARE THERE? ANSWER: 72.

OBSERVE THAT THE ISOMORPHISM RELATION AMONG GRAPHS IS AN EQUIVALENCE RELATION.

EXERCISE:

VERIFY THIS.

THE EQUIVALENCE CLASSES OF THIS RELATION ARE SOMETIMES CALLED <u>ISOMORPHISM CLASSES</u>.

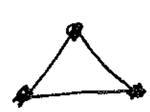
THUS GRAPH THEORY IS REALLY THE STUDY OF GRAPH ISOMORPHISM CLASSES.

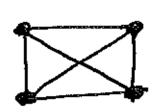
<u>DEFN</u>.
A <u>COMPLETE GRAPH</u> IS A SIMPLE GRAPH IN WHICH EACH PAIR OF DISTINCT VERTICES ARE JOINED BY AN EDGE.

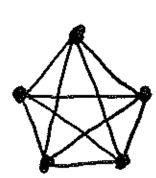
UP TO ISOMORPHISM, THERE IS JUST ONE COMPLETE GRAPH ON n VERTICES, WHICH WE DENOTE BY K_n

K_1 0 EDGES

K_2  1 EDGE

K_3  3 EDGES

K_4  6 EDGES

K_5  10 EDGES

HOW MANY EDGES HAS K_n ?

LET $V(K_n) = \{1, 2, \ldots, n\}$. THEN THE EDGES OF K_n ARE IN ONE TO ONE CORRESPONDENCE WITH THE UNORDERED PAIRS OF DISTINCT VERTICES OF K_n. i.e. THERE IS A BIJECTION

$$E(K_n) \longrightarrow \{\text{2-ELEMENT SUBSETS OF } V(K_n)\}$$

THUS $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$.

DEFN.

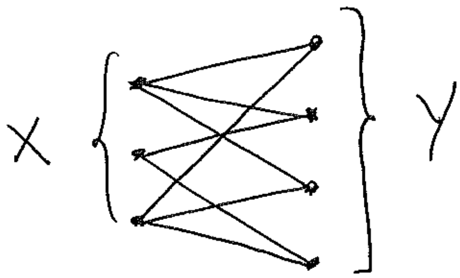
A GRAPH IS CALLED NULL (OR EMPTY OR TRIVIAL) IF $E(G) = \emptyset$. THE NULL GRAPH ON n VERTICES IS SOMETIMES DENOTED N_n.

DEFN.

A GRAPH G IS CALLED BIPARTITE IF $V(G)$ CAN BE PARTITIONED INTO TWO NONEMPTY SUBSETS X, Y (i.e. $X \cup Y = V$, $X \cap Y = \emptyset$) SUCH THAT EACH EDGE OF G HAS ONE END IN X AND ONE END IN Y. (i.e. NO EDGE JOINS X TO X OR Y TO Y).

THE PAIR X, Y IS CALLED A BIPARTITION OF $V(G)$.

EX

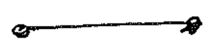


DEFN:

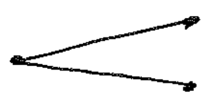
A <u>COMPLETE BIPARTITE GRAPH</u> IS A SIMPLE BIPARTITE GRAPH IN WHICH EACH VERTEX IN X IS JOINED TO EACH VERTEX IN Y BY AN EDGE.

UP TO ISOMORPHISM THERE IS JUST ONE COMPLETE BIPARTITE GRAPH WITH $|X| = n$ AND $|Y| = m$, WHICH WE DENOTE $K_{n,m}$

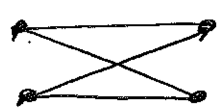
$K_{1,1}$



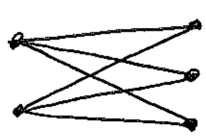
$K_{1,2}$



$K_{2,2}$

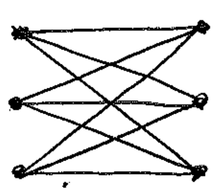


$K_{2,3}$



$\boxed{\text{OBSERVE THAT } |E(K_{n,m})| = nm}$

$K_{3,3}$



$\vdots$