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The Definition, Determination, and Characterization of Acceleration Sets for Spatial Manipulators

Abstract

In this article, the approach developed by the authors for systematically studying the acceleration capability and properties of the end effector of a planar manipulator is extended to the general, serial, spatial manipulator possessing three degrees of freedom. The acceleration of the end effector at a given configuration of the manipulator is a linear function of the actuator torques and a (nonlinear) quadratic function of the joint velocities. By decomposing the functional relationships between the inputs (actuator torques and joint velocities) and the output (acceleration of the end effector) into two fundamental mappings—a linear mapping between the actuator torques and the acceleration space of the end effector and a quadratic (nonlinear) mapping between the joint velocities and the acceleration space of the end effector—and by deriving the properties of these two mappings, it is possible to determine the properties of all acceleration sets that are the images of the appropriate input sets under the two fundamental mappings. A central feature of this article is the determination of the properties of the quadratic mapping, which then makes it possible to obtain analytic expressions for most acceleration properties of interest. We show that a fundamental way of studying these quadratic mappings is in terms of the mapping of (input) line congruences into (output) line congruences. The article concludes with the application of the analytical results to the computation of the various acceleration properties of an actual spatial manipulator.

1. Introduction

In this article, we apply the approach developed in Desa and Kim (1990a) to the problem of determining the acceleration capability and acceleration properties of a reference point on the end effector of a spatial three-degree-of-freedom manipulator.

An informal statement of the problem is as follows:

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Consider the general three-degree-of-freedom revolute-joint manipulator shown schematically in Figure 1. We are interested in studying the acceleration of a reference point P on link 3. P is typically a point on the joint axis of the end effector; the acceleration of P is therefore often referred to as the end-effector acceleration. The usefulness of studying the acceleration of the end effector has been discussed in Yoshikawa (1985), Khatib and Burdick (1987), Graettinger and Krogh (1988), Desa and Kim (1990b), and Kim (1989).

As shown, for example, in Desa and Kim (1990a), the acceleration capability of the point P under various conditions is best described by certain acceleration sets. Two properties that are used, in general, to characterize these sets are the maximum possible magnitude of the acceleration of P and the maximum magnitude of the acceleration of P that is available in all directions. The former prop-

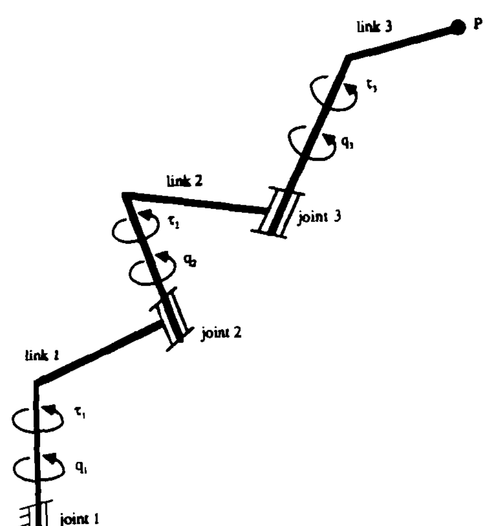

Fig. 1. Schematic diagram of a three-degree-of-freedom spatial manipulator.

erty is simply called the maximum acceleration of P and the latter the isotropic acceleration of P (Khatib and Burdick 1987).

Acceleration properties of the end effector have also been studied by Yoshikawa (1985), Khatib and Burdick (1987), and Graettinger and Krogh (1988). The approach of each of these researchers has been discussed and compared with our approach in Desa and Kim (1990b), and we will not repeat that discussion here. We will, however, repeat the following fundamental hypothesis underlying our approach:

By decomposing the functional relationships between the inputs (actuator torques and joint velocities) and the output (acceleration of P) into two fundamental mappings—a linear mapping between actuator torque space and the acceleration space of point P and a quadratic (nonlinear) mapping between the "joint velocity" space and the acceleration space of P—and by deriving the properties of these two mappings, it is possible to determine the properties of all acceleration sets that are the images of the appropriate input sets under the two fundamental mappings.

The contributions of this article are as follows:

1. The central contribution of this article is the determination of the properties of the quadratic mapping between the joint velocity space and the acceleration space of P, which then makes it possible to obtain analytical solutions for the isotropic acceleration. We show that a fundamental way of developing the properties of the quadratic mappings of interest is in terms of the mapping of (input) line congruences into (output) line congruences.
2. Closed-form analytic expressions are obtained for relating important acceleration properties of manipulators to all the manipulator parameters and input variables (torques, joint velocities) of interest. (The only exception is the maximum local acceleration, which is specified in terms of tight lower and upper bounds in Section 5.4.)
3. Necessary and sufficient conditions for the existence of isotropic acceleration have been determined. (Earlier studies seem to implicitly assume that isotropic acceleration always exists.) These conditions are stated explicitly in terms of manipulator parameters and input variables.
4. Analytic expressions are derived for determining the maximum and isotropic acceleration of the end effector at any ("local") configuration of the manipulator.

We will present detailed derivations of all results pertaining to the quadratic mapping and the corresponding acceleration set, since these results play an important role in the development of acceleration set theory. Detailed proofs of all the other results given below can be found in Kim (1989).

In Section 6, we will demonstrate the application of the theory to a particular three-degree-of-freedom spatial manipulator. (The application of acceleration theory to problems in manipulator design has been dealt with in Desa and Kim [1990b].) The next section, which describes our approach, also provides a "road map" of the article.

2. Description of the Approach

The approach for studying the acceleration of a reference point P on the end effector, given in Desa and Kim (1990a) is as follows:

1. Define the input variables and output variables of interest (Section 3.1.). The output of interest is the acceleration of the reference point P.
2. Define the input sets of interest (Section 3.1).
3. Define the input-output functional relations. These are obtained from the dynamical and kinematical equations of the manipulator (Section 3.2).
4. Define fundamental mappings from these functional relations (Section 3.3). There are two fundamental mappings: a linear mapping and a quadratic mapping.
5. Define the image sets of the input sets under the mappings obtained in step 4 (Section 3.4). These image sets are the acceleration sets of interest.
6. Define general properties that can be used to characterize ("measure") acceleration sets (Section 3.5).
7. Determine the properties of the mappings defined in step 4 (Section 4).
8. Determine the acceleration sets defined in step 5 using the properties of the mappings obtained in step 7 (Section 4).
9. Determine expressions for the properties of the acceleration sets determined in step 8 (Section 5).
10. Determine the local acceleration properties for any configuration $\mathbf{q}$ of the manipulator using the properties of the acceleration sets obtained in step 9 (Section 5.4).

3. Definition of the Acceleration Sets

3.1. Manipulator Input and Output Variables

Consider the spatial three-degree-of-freedom manipulator with three revolute joints shown schematically in Figure 1. In this section, we define the link parameters, the input variables, the input sets, the output variables, and the output sets for this general spatial manipulator. The manipulator is assumed to be rigid with negligible joint friction. The manipulator can be described by a set of geometric and inertia parameters, which will depend on the manipulator type (Kim 1989). Next, we define the input variables, the input constraints, and the corresponding input sets of the three-degree-of-freedom spatial manipulator. Let q_1, q_2, and q_3 denote the generalized coordinates of the manipulator (see Fig. 9): q_1, q_2, and q_3 being the joint variables, respectively, at joints 1, 2, 3. Define

$$\mathbf{q} \triangleq \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{1}$$

to be a vector of joint variables; the corresponding vector space of all $\mathbf{q}$ is called the joint space. If

$$q_{iL} \leq q_i \leq q_{iU}, (i = 1, 2, 3) \tag{2}$$

represents the constraint on joint variable i, the workspace W of a manipulator is defined as

$$W = \{\mathbf{q} | q_{iL} \leq q_i \leq q_{iU}, i = 1, 2, 3\}. \tag{3}$$

Let $\dot{q}_1, \dot{q}_2$, and $\dot{q}_3$ denote the joint velocities. Define

$$\dot{\mathbf{q}} \triangleq \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \tag{4}$$

to be the vector of the joint velocities. If

$$|\dot{q}_i| \leq \dot{q}_{i0}, (i = 1, 2, 3) \tag{5}$$

denotes the constraints on the joint variable rates, then we can define

$$F = \{\dot{\mathbf{q}} |\ |\dot{q}_i| \leq \dot{q}_{i0}, i = 1, 2, 3\} \tag{6}$$

to be the set of all the possible joint velocity vectors represented by the regular parallelopiped $J_1 K_1 L_1 M_1 J_2 K_2 \times L_2 M_1$ in Figure 2. (We will refer to this parallelopiped as the parallelopiped F for short.)

Let τ_1, τ_2, and τ_3 denote the actuator torques, respectively, at joints 1, 2, and 3, and

$$\tau \triangleq \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \tag{7}$$

denotes the vector of actuator torque vectors. Let

$$|\tau_i| \leq \tau_{i0}, (i = 1, 2, 3) \tag{8}$$

denote the constraints on the actuator torques at joints 1, 2, and 3. Define

$$T = \{\tau |\ |\tau_i| \leq \tau_{i0}, i = 1, 2, 3\} \tag{9}$$

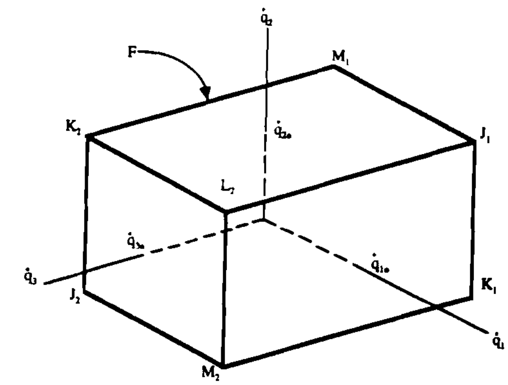

Fig. 2. The set F of the joint velocities for a three-degree-of-freedom spatial manipulator.

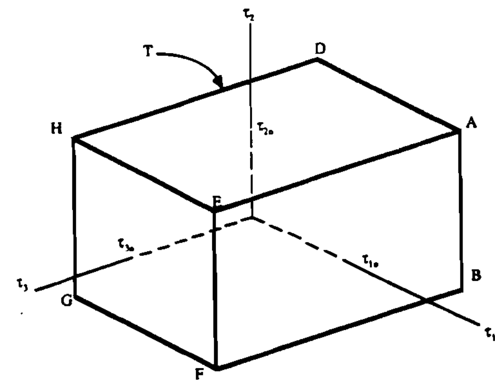

Fig. 3. The set T of actuator torques for a three-degree-of-freedom spatial manipulator.

as the set of the allowable actuator torques, represented by regular parallelopiped $ABCDEFGH$ in Figure 3. (We will refer to this parallelopiped as the parallelopiped T for short.)

The vectors $\mathbf{q}$, $\dot{\mathbf{q}}$ and τ will be referred to as the input variables (more precisely, the input variable vectors) of the manipulator. We will also refer to the vector $\mathbf{q}$ as a configuration of the manipulator.

Let (x_1, x_2, x_3) denote the coordinates, in a reference frame fixed to the base, of a reference point P on link 3 (see Fig. 1) and define

$$\mathbf{x}^p \triangleq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{10}$$

as the vector of task coordinates; the corresponding vector space of all $\mathbf{x}^p$ is called the task space.

The velocity $\dot{\mathbf{x}}^p$ and the acceleration $\ddot{\mathbf{x}}^p$ of the point P of the manipulator are, respectively, given by

$$\dot{\mathbf{x}}^p = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \tag{11}$$

and

$$\ddot{\mathbf{x}}^p = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}. \quad (12)$$

The acceleration of P, $\ddot{\mathbf{x}}^p$, is the output variable of interest in the present work. The corresponding vector space A of all possible $\ddot{\mathbf{x}}^p$ is called the acceleration space, expressed by

$$A = \{\ddot{\mathbf{x}} | \ddot{\mathbf{x}} \in R^3\}. \quad (13)$$

3.2. Functional Relations Between the Inputs $\dot{\mathbf{q}}$, τ and the Acceleration $\ddot{\mathbf{x}}^p$

The next step is to obtain the functional relations between the acceleration $\ddot{\mathbf{x}}^p$ and the inputs $\dot{\mathbf{q}}$ and $\boldsymbol{\tau}$ for a given configuration $\mathbf{q}$. In this section, we show how the necessary functional relations can be obtained from the manipulator dynamical equations and the so-called manipulator Jacobian relationship.

The dynamic behavior of the most general three-degree-of-freedom rigid spatial manipulator (Fig. 1) can be written in the following symbolic form (Craig 1986):

$$\mathbf{D}\ddot{\mathbf{q}} + \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{p} = \tau, \quad (14)$$

where $\mathbf{D}$ is the so-called mass matrix of the manipulator, $\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}})$ is the vector consisting of all terms that are nonlinear in the products of the joint variable rates $\dot{q}_i$, $(i = 1, 2, 3)$, and $\mathbf{p}$ is a vector of all terms due to gravity.

We next express nonlinear terms $\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}})$ as products of a matrix and a vector. To understand how this is done, we first write $\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}})$ in its most general expanded form,

$$\mathbf{V} = \begin{bmatrix} u_{11}\dot{q}_1^2 + u_{12}\dot{q}_2^2 + u_{13}\dot{q}_3^2 + 2w_{11}\dot{q}_1\dot{q}_2 \\ \quad +2w_{12}\dot{q}_2\dot{q}_3 + 2w_{13}\dot{q}_3\dot{q}_1 \\ u_{21}\dot{q}_1^2 + u_{22}\dot{q}_2^2 + u_{23}\dot{q}_3^2 + 2w_{21}\dot{q}_1\dot{q}_2 \\ \quad +2w_{22}\dot{q}_2\dot{q}_3 + 2w_{23}\dot{q}_3\dot{q}_1 \\ u_{31}\dot{q}_1^2 + u_{32}\dot{q}_2^2 + u_{33}\dot{q}_3^2 + 2w_{31}\dot{q}_1\dot{q}_2 \\ \quad +2w_{32}\dot{q}_2\dot{q}_3 + 2w_{33}\dot{q}_3\dot{q}_1 \end{bmatrix}. \quad (15)$$

Defining the two matrix operators,

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \quad (16)$$

and

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix}, \quad (17)$$

and two vector operators,

$$\langle\dot{\mathbf{q}}\rangle^2 = \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \dot{q}_3^2 \end{bmatrix} \quad (18)$$

and

$$[\dot{\mathbf{q}}]^2 = \begin{bmatrix} 2\dot{q}_1\dot{q}_2 \\ 2\dot{q}_2\dot{q}_3 \\ 2\dot{q}_1\dot{q}_1 \end{bmatrix}, \quad (19)$$

we can decompose the nonlinear term $\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}})$ as follows:

$$\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \dot{q}_3^2 \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} \begin{bmatrix} 2\dot{q}_1\dot{q}_2 \\ 2\dot{q}_2\dot{q}_3 \\ 2\dot{q}_3\dot{q}_1 \end{bmatrix}. \quad (20)$$

$$= \mathbf{U}\langle\dot{\mathbf{q}}\rangle^2 + \mathbf{W}[\dot{\mathbf{q}}]^2. \quad (21)$$

Substituting equation (21) into (14), we can express the dynamic equation of a general spatial manipulator by

$$\mathbf{D}\ddot{\mathbf{q}} + \mathbf{U}\langle\dot{\mathbf{q}}\rangle^2 + \mathbf{W}[\dot{\mathbf{q}}]^2 + \mathbf{p} = \boldsymbol{\tau}. \quad (22)$$

This is the most general expression describing the dynamics of a rigid three-degree-of-freedom spatial manipulator in the joint space.

The relationship between the velocity, $\dot{\mathbf{x}}^p$, of point P, and the joint velocity vector $\dot{\mathbf{q}}$ is well known (Desa and Roth 1985):

$$\dot{\mathbf{x}}^p = \mathbf{J}\dot{\mathbf{q}}, \quad (23)$$

where $\mathbf{J}$ is a (3×3) matrix called the manipulator Jacobian. The detailed expression of Jacobian matrix is given in the Appendix.

To obtain the expression for the acceleration $\ddot{\mathbf{x}}^p$ of the point P, we differentiate equation (23),

$$\ddot{\mathbf{x}}^p = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}}. \quad (24)$$

The second term in equation (24), $\dot{\mathbf{J}}\dot{\mathbf{q}}$, can be written in the form (Kim 1989):

$$\dot{\mathbf{J}}\dot{\mathbf{q}} = -\mathbf{F}\langle\dot{\mathbf{q}}\rangle^2 - \mathbf{G}[\dot{\mathbf{q}}]^2, \quad (25)$$

where $\mathbf{F}$ and $\mathbf{G}$ are appropriately defined 3×3 matrices.

Substituting equations (25) into (24), we obtain

$$\ddot{\mathbf{x}}^p = \mathbf{J}\ddot{\mathbf{q}} - \mathbf{F}\langle\dot{\mathbf{q}}\rangle^2 - \mathbf{G}[\dot{\mathbf{q}}]^2. \quad (26)$$

Defining the quantities,

$$\mathbf{A} = \mathbf{J}\mathbf{D}^{-1}, \quad (27)$$

$$\mathbf{B} = -\mathbf{A}\mathbf{U} - \mathbf{F}, \quad (28)$$

$$\mathbf{N} = -\mathbf{A}\mathbf{W} - \mathbf{G}, \quad (29)^*$$

and

$$\mathbf{s} = -\mathbf{A}\mathbf{p}, \quad (31)$$

we can easily show that the acceleration $\ddot{\mathbf{x}}^p$ of point P, obtained by combining equation (22) with equations (26)–(29) and (31), is given by

$$\ddot{\mathbf{x}}^p = \mathbf{A}\boldsymbol{\tau} + \mathbf{B}\langle\dot{\mathbf{q}}\rangle^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}, \quad (32)$$

* Note: There is no eq. (30).

where $\mathbf{A}, \mathbf{B}, \mathbf{N}, \mathbf{s}$ are configuration dependent and have the components $a_{ij}, b_{ij}, n_{ij}, s_i$, $(i, j = 1, 2, 3)$.

Equation (32) expresses the required (input-output) functional relation between the input variables, $\dot{\mathbf{q}}$ and $\boldsymbol{\tau}$, and the acceleration $\ddot{\mathbf{x}}^P$ of the point P (the output variable) at a given configuration $\mathbf{q}$. It is important to note that the definition of the matrix "operators" $\mathbf{U}, \mathbf{W}, \mathbf{F}$, and $\mathbf{G}$ and the vectors $\langle\dot{\mathbf{q}}\rangle^2$ and $[\dot{\mathbf{q}}]^2$ enables us to write the dynamic equations in the compact form (32), which is critical in the sequel.

3.3. Mappings

In this section, we define two fundamental mappings between the input variables and the acceleration $\ddot{\mathbf{x}}^P$ of the point P (the output variable).

It is convenient to regard the functional relation (32) as a mapping between the input variables $\dot{\mathbf{q}}$ and $\boldsymbol{\tau}$ and the output variable $\ddot{\mathbf{x}}^P$ for a given configuration $\mathbf{q}$ of the manipulator. Furthermore, defining

$$\ddot{\mathbf{x}}^P_\tau \triangleq \begin{bmatrix} \alpha_{1\tau} \\ \alpha_{2\tau} \\ \alpha_{3\tau} \end{bmatrix} = \mathbf{A}\boldsymbol{\tau} \qquad (33)$$

and

$$\ddot{\mathbf{x}}^P_{\dot{\mathbf{q}}} \triangleq \begin{bmatrix} \alpha_{1\dot{\mathbf{q}}} \\ \alpha_{2\dot{\mathbf{q}}} \\ \alpha_{3\dot{\mathbf{q}}} \end{bmatrix} = \mathbf{B}\langle\dot{\mathbf{q}}\rangle^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}, \qquad (34)$$

equation (32) can be written as

$$\ddot{\mathbf{x}}^P = \ddot{\mathbf{x}}^P_\tau + \ddot{\mathbf{x}}^P_{\dot{\mathbf{q}}}. \qquad (35)$$

It is convenient to regard the vector $\ddot{\mathbf{x}}^P_\tau$ as the contribution of the joint torques to the acceleration of the reference point P, and the vector $\ddot{\mathbf{x}}^P_{\dot{\mathbf{q}}}$ as the contribution of the joint velocities and gravity to the acceleration of P. Equation (35) expresses the fact that the sum of these two vectors gives us the acceleration of the end effector P of the manipulator.

Equation (33) can be viewed as a linear, configuration-dependent mapping between the torque vector $\boldsymbol{\tau}$ and its contribution $\ddot{\mathbf{x}}^P_\tau$ to the acceleration of P. Similarly, equation (34) can be viewed as a quadratic, configuration-dependent mapping between the joint velocity vector $\dot{\mathbf{q}}$ and its contribution $\ddot{\mathbf{x}}^P_{\dot{\mathbf{q}}}$ to the acceleration of P for a given configuration $\mathbf{q}$. These are the two mappings of interest in the sequel.

3.4. Manipulator Acceleration Sets

Having defined two fundamental mappings of interest, we are interested in the image sets of the input sets under the mappings (33) and (34) at a given configuration $\mathbf{q}$ of the manipulator. There are three image sets of interest.

3.4.1. Image Set S_τ of the Actuator Torque Set T Under the Linear Mapping

For a given set T of the actuator torques τ described by equation (9) and represented graphically by a regular parallelopiped in the τ-space (see Fig. 3), we define the image set S_τ of T under the linear mapping (33) as

$$S_\tau = \{\ddot{\mathbf{x}}^P_\tau | \ddot{\mathbf{x}}^P_\tau = \mathbf{A}\tau, \tau \in T\}. \qquad (36)$$

(Note that S_τ lies in the acceleration space A.)

3.4.2. Image Set $S_{\dot{\mathbf{q}}}$ of the Joint Variable Rate Set F Under the Quadratic Mapping

For a given set F of the joint velocities $\dot{\mathbf{q}}$ described by equation (6) and represented graphically by a regular parallelopiped (see Fig. 2), we define the image set $S_{\dot{\mathbf{q}}}$ of F under the quadratic mapping as

$$S_{\dot{\mathbf{q}}} = \{\ddot{\mathbf{x}}^P_{\dot{\mathbf{q}}} | \ddot{\mathbf{x}}^P_{\dot{\mathbf{q}}} = \mathbf{B}\langle\dot{\mathbf{q}}\rangle^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}, \dot{\mathbf{q}} \in F\}. \qquad (37)$$

(Note that $S_{\dot{\mathbf{q}}}$ lies in the acceleration space A.) From equation (32) and the above definition (37), we see that the image set $S_{\dot{\mathbf{q}}}$ represents the set of all possible accelerations (the acceleration capability of the manipulator) when the actuators are turned off ($\tau = 0$) in any configuration $\mathbf{q}$.

3.4.3. State Acceleration Set

When a manipulator is in motion, the dynamic state of a manipulator can be specified by the joint variables, (q_1, q_2), and joint velocities $(\dot{q}_1, \dot{q}_2)$. The state vector $\mathbf{u}$ that characterizes the dynamic state of the manipulator is defined as follows:

$$\mathbf{u} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}. \qquad (38)$$

For a specified dynamic state of a three-degree-of-freedom manipulator, the last three terms of the acceleration $\ddot{\mathbf{x}}^P$ in equation (32) are constant vectors that we denote by $\mathbf{k(u)}$ and define as follows:

$$\mathbf{k(u)} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \mathbf{B}\langle\dot{\mathbf{q}}\rangle^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}. \qquad (39)$$

Equation (32) can then be written as follows:

$$\ddot{\mathbf{x}} = \mathbf{A}\boldsymbol{\tau} + \mathbf{k}. \qquad (40)$$

For a given dynamic state $\mathbf{u}$ of the manipulator, we define the state acceleration set $S_\mathbf{u}$ as the image set of T under the linear mapping (40):

$$S_\mathbf{u} = \{\ddot{\mathbf{x}}^P | \ddot{\mathbf{x}}^P = \mathbf{A}\tau + \mathbf{k}, \tau \in T\}. \qquad (41)$$

$S_{\mathbf{u}}$ is therefore the set of all possible accelerations at any given dynamic state $\mathbf{u}$ of the manipulator. Because the dynamic state $\mathbf{u}$ of the manipulator essentially specifies the velocity $\dot{\mathbf{x}}^P$ of the point P (see equation (23)) in any configuration, we can also interpret the state acceleration set $S_{\mathbf{u}}$ (the set of available accelerations) as the acceleration capability of the manipulator when the manipulator is moving with the velocity $\dot{\mathbf{x}}^P$ in a given configuration $\mathbf{q}$.

3.5. Properties of the Acceleration Sets

The definitions of the acceleration sets in the previous subsection will be used in Section 5 to determine these sets. Once these sets have been determined, one would like to characterize them.

Consider an acceleration set S in the acceleration space $\ddot{\mathbf{x}}$ and two spheres C_1 and C_2: C_1 is the smallest sphere centered at the origin that completely encloses the acceleration set, and C_2 is the largest sphere centered at the origin that lies inside the acceleration set. The radius r_1 of the sphere C_1 is the maximum magnitude of the available acceleration in S. The radius r_2 of sphere 2 represents the maximum magnitude of the acceleration available in all directions.

We therefore define the following two properties of S:

- The maximum acceleration of S: $a_{\text{max}}(S) = r_1$,
- The isotropic acceleration of S: $a_{\text{iso}}(S) = r_2$.

Comments: The isotropic and maximum acceleration are particularly attractive for characterizing set S, in contrast to the average acceleration, as they can be readily extracted from the dynamic equations in closed form or by appropriate bounds. The average acceleration, if required, can be numerically determined from the description of the acceleration sets given in the next section.

4. Determination of the Acceleration Sets

Analytic expressions for the determination of the three sets S_τ, $S_{\dot{\mathbf{q}}}$, and $S_{\mathbf{u}}$ are presented, respectively, in Sections 4.1, 4.2, and 4.3. The determination of S_τ and the state acceleration set $S_{\mathbf{u}}$ follows directly from well-known properties of linear mappings, while the determination of the set $S_{\dot{\mathbf{q}}}$ requires the derivation of the properties of quadratic mappings that are new.

4.1. Determination of the Image Set S_τ

The set S_τ is the image set of the actuator torque set T under the linear mapping (33). We determine the image set S_τ of the linear mapping of a three-degree-of-freedom manipulator in the $\ddot{\mathbf{x}}$ space. Additionally, we identify the boundaries of set S_τ, which are planes in the $\ddot{\mathbf{x}}$ space.

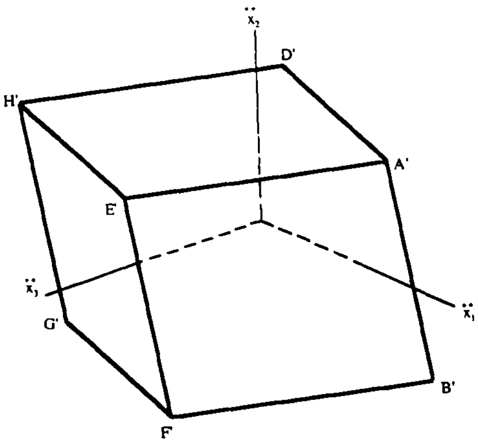

Fig. 4. The image set $\mathbf{S}_\tau$ *of a three-degree-of-freedom spatial manipulator.*

Result 1: The image set S_τ of the actuator torque set T under the linear mapping (34) is the interior and boundary of the parellelopiped $A'B'C'D'E'F'G'H'$ (Fig. 4) in the $\ddot{\mathbf{x}}$ space whose vertices $A', B', \ldots, H'$ are as follows:

$$A' : (a_{11}\tau_{10} + a_{12}\tau_{20} + a_{13}\tau_{30},\ a_{21}\tau_{10} + a_{22}\tau_{20} + a_{23}\tau_{30},\ a_{31}\tau_{10} + a_{32}\tau_{20} + a_{33}\tau_{30}) \quad (42)$$

$$B' : (a_{11}\tau_{10} - a_{12}\tau_{20} + a_{13}\tau_{30},\ a_{21}\tau_{10} - a_{22}\tau_{20} + a_{23}\tau_{30},\ a_{31}\tau_{10} - a_{32}\tau_{20} + a_{33}\tau_{30}) \quad (43)$$

$$C' : (-a_{11}\tau_{10} - a_{12}\tau_{20} + a_{13}\tau_{30},\ -a_{21}\tau_{10} - a_{22}\tau_{20} + a_{23}\tau_{30},\ -a_{31}\tau_{10} - a_{32}\tau_{20} + a_{33}\tau_{30}) \quad (44)$$

$$D' : (-a_{11}\tau_{10} + a_{12}\tau_{20} + a_{13}\tau_{30},\ -a_{21}\tau_{10} + a_{22}\tau_{20} + a_{23}\tau_{30},\ -a_{31}\tau_{10} + a_{32}\tau_{20} + a_{33}\tau_{30}) \quad (45)$$

$$E' : (a_{11}\tau_{10} + a_{12}\tau_{20} - a_{13}\tau_{30},\ a_{21}\tau_{10} + a_{22}\tau_{20} - a_{23}\tau_{30},\ a_{31}\tau_{10} + a_{32}\tau_{20} - a_{33}\tau_{30}) \quad (46)$$

$$F' : (a_{11}\tau_{10} - a_{12}\tau_{20} - a_{13}\tau_{30},\ a_{21}\tau_{10} - a_{22}\tau_{20} - a_{23}\tau_{30},\ a_{31}\tau_{10} - a_{32}\tau_{20} - a_{33}\tau_{30}) \quad (47)$$

$$G' : (-a_{11}\tau_{10} - a_{12}\tau_{20} - a_{13}\tau_{30},\ -a_{21}\tau_{10} - a_{22}\tau_{20} - a_{23}\tau_{30},\ -a_{31}\tau_{10} - a_{32}\tau_{20} - a_{33}\tau_{30}) \quad (48)$$

$$H' : (-a_{11}\tau_{10} + a_{12}\tau_{20} - a_{13}\tau_{30},\ -a_{21}\tau_{10} + a_{22}\tau_{20} - a_{23}\tau_{30},\ -a_{31}\tau_{10} + a_{32}\tau_{20} - a_{33}\tau_{30}) \quad (49)$$

where a_{ij}, $(i, j = 1, 2, 3)$, are the elements of the matrix $\mathbf{A}$. The centroid of the parallelopiped $A'B' \ldots H'$ (see Fig. 4) is the origin of the $\ddot{\mathbf{x}}$ plane.

Result 2: The planar sides of the parallelopiped S_τ are given by the following equations:

$$A'B'F'E' : (a_{22}a_{33} - a_{23}a_{32})\ddot{x}_1 - (a_{12}a_{33} - a_{32}a_{13})\ddot{x}_2 + (a_{12}a_{23} - a_{13}a_{22})\ddot{x}_3 = \tau_{10}\det(\mathbf{A}) \quad (50)$$

$$D'C'G'H' : (a_{22}a_{33} - a_{23}a_{32})\ddot{x}_1 - (a_{12}a_{33} - a_{32}a_{13})\ddot{x}_2 + (a_{12}a_{23} - a_{13}a_{22})\ddot{x}_3 = -\tau_{10}\det(\mathbf{A}) \quad (51)$$

$$A'D'H'E' : -(a_{21}a_{33} - a_{23}a_{31})\ddot{x}_1 + (a_{11}a_{33} - a_{31}a_{13})\ddot{x}_2 - (a_{11}a_{23} - a_{13}a_{21})\ddot{x}_3 = \tau_{20}\det(\mathbf{A}) \quad (52)$$

$$B'C'G'F' : (a_{21}a_{33} - a_{23}a_{31})\ddot{x}_1 - (a_{11}a_{33} - a_{31}a_{13})\ddot{x}_2 + (a_{11}a_{23} - a_{13}a_{21})\ddot{x}_3 = \tau_{20}\det(\mathbf{A}) \quad (53)$$

$$A'B'C'D' : (a_{21}a_{32} - a_{22}a_{31})\ddot{x}_1 - (a_{11}a_{32} - a_{31}a_{12})\ddot{x}_2 + (a_{11}a_{22} - a_{12}a_{21})\ddot{x}_3 = \tau_{30}\det(\mathbf{A}) \quad (54)$$

$$E'F'G'H' : (a_{21}a_{32} - a_{22}a_{31})\ddot{x}_1 - (a_{11}a_{32} - a_{31}a_{12})\ddot{x}_2 + (a_{11}a_{22} - a_{12}a_{21})\ddot{x}_3 = -\tau_{30}\det(\mathbf{A}) \quad (55)$$

where det(**A**) is the determinant of the matrix **A**.

The following are well-known properties of a linear mapping that are useful in the sequel:

1. A plane in the τ-space will map into a plane in the $\ddot{\mathbf{x}}$-plane. In particular, planes $p_1(\tau_1 = 0), p_2(\tau_2 = 0)$, and $p_3(\tau_3 = 0)$ map, respectively, into planes p'_1, p'_2, and p'_3 whose equations are as follows:

$$p'_1 : (a_{22}a_{33} - a_{23}a_{32})\ddot{x}_1 - (a_{12}a_{33} - a_{32}a_{13})\ddot{x}_2 + (a_{12}a_{23} - a_{13}a_{22})\ddot{x}_3 = 0, \quad (56)$$

$$p'_2 : (a_{21}a_{33} - a_{23}a_{31})\ddot{x}_1 - (a_{11}a_{33} - a_{31}a_{13})\ddot{x}_2 + (a_{11}a_{23} - a_{13}a_{21})\ddot{x}_3 = 0, \quad (57)$$

$$p'_3 : (a_{21}a_{32} - a_{22}a_{31})\ddot{x}_1 - (a_{11}a_{32} - a_{31}a_{12})\ddot{x}_2 + (a_{11}a_{22} - a_{12}a_{21})\ddot{x}_3 = 0. \quad (58)$$

   (All three planes p'_1, p'_2, and p'_3 pass through the origin of the $\ddot{\mathbf{x}}$ plane.)
2. Any plane g_1 parallel to p_1 maps into a plane g'_1 parallel to p'_1.
3. Any plane g_2 parallel to p_2 maps into a plane g'_2 parallel to p'_2.
4. Any plane g_3 parallel to p_3 maps into a plane g'_3 parallel to p'_3.

4.2. Determination of the Image Set $S_{\dot{\mathbf{q}}}$

The set $S_{\dot{\mathbf{q}}}$ for a spatial three-degree-of-freedom manipulator is the image set of the joint rate set F under mapping (34). We decompose the set F (Fig. 5A) into three subsets F_1, F_2, and F_3 described as follows:

DEFINITION 1. The set F_1 is the truncated line congruence (Semple and Kneebone 1952) consisting of the doubly infinite set of line segments passing through the origin with one end point on the plane $J_1K_1M_2L_2$ and the other end point on the plane $M_1L_1J_2K_2$. A typical member of F_1 is the line segment g_1 shown in Figure 5B.

DEFINITION 2. The set F_2 is the truncated line congruence consisting of the doubly infinite set of line segments passing through the origin with one end point on the plane $J_1L_2K_2M_1$ and the other end point on the plane $K_1M_2J_2L_1$. A typical member of F_2 is the line segment g_2 shown in Figure 5C.

DEFINITION 3. The set F_3 is the truncated line congruence consisting of the doubly infinite set of line segments passing through the origin with one end point on the plane $J_1K_1L_1M_1$ and the other end point on the plane $L_2M_2J_2K_2$. A typical member of F_3 is the line segment g_3 shown in Figure 5D.

We can now state the useful results that analytically describe $S_{\dot{\mathbf{q}}}$, the image of F.

Result 1:

1(a). Every line of the type g_1 belonging to set F_1 maps into a line g'_1 in the $\ddot{\mathbf{x}}$ space (Fig. 6A), one end point of which is the point S whose coordinates s_i, $(i = 1, 2, 3)$, are given by (31) and the other end point of which lies on the quadratic surface patch (Fig. 6B) whose parametric equation (in $\dot{q}_2$ and $\dot{q}_3$) is:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11}\dot{q}_{10}^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_{10}\dot{q}_2 \\ \quad + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_{10} + s_1 \\ b_{21}\dot{q}_{10}^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_{10}\dot{q}_2 \\ \quad + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_{10} + s_2 \\ b_{31}\dot{q}_{10}^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_{10}\dot{q}_2 \\ \quad + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_{10} + s_3 \end{bmatrix}. \quad (59)$$

where

$$|\dot{q}_2| < \dot{q}_{20}, \quad |\dot{q}_3| < \dot{q}_{30}.$$

1(b). The set F_1 maps into a set $(S_{\dot{\mathbf{q}}})_1$ in the $\ddot{\mathbf{x}}$ plane that is a doubly infinite system of line segments, one end point of which is the point S with coordinates $s_i, (i = 1, 2, 3)$, given by (31), and the other end point of which lies on the quadratic surface patch described by (59).

Result 2:

2(a). Every line of the type g_2 belonging to the set F_2 maps into a line g'_2 in the $\ddot{\mathbf{x}}$ space (Fig. 6C), one end point of which is the point S and the other end point of which lies on the quadratic surface patch (Fig. 6D) whose parametric equation (in $\dot{q}_3$ and $\dot{q}_1$) is:

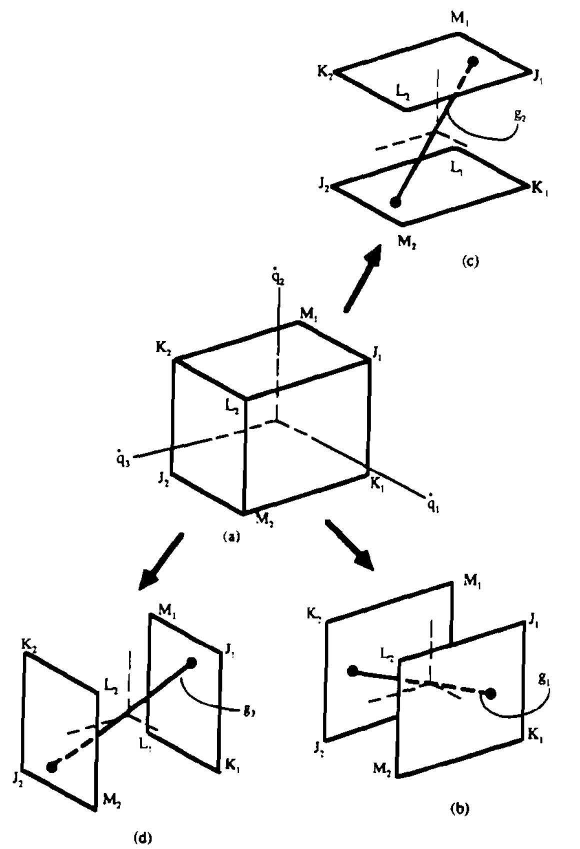

Fig. 5. The image set $\mathbf{S}_{\dot{\mathbf{q}}}$ of a three-degree-of-freedom spatial manipulator.

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11}\dot{q}_1^2 + b_{12}\dot{q}_{20}^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_{20} \\ \quad + 2n_{12}\dot{q}_{20}\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1 \\ b_{21}\dot{q}_1^2 + b_{22}\dot{q}_{20}^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_{20} \\ \quad + 2n_{22}\dot{q}_{20}\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2 \\ b_{31}\dot{q}_1^2 + b_{32}\dot{q}_{20}^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_{20} \\ \quad + 2n_{32}\dot{q}_{20}\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3 \end{bmatrix}. \quad (60)$$

where

$$|\dot{q}_1| < \dot{q}_{10}, \quad |\dot{q}_3| < \dot{q}_{30}.$$

2(b). The set F_2 maps into a set $(S_{\dot{\mathbf{q}}})_2$ in the $\ddot{\mathbf{x}}$ plane that is a doubly infinite system of line segments, one end point of which is the point S and the other end point of which lies on the quadratic surface described by (60).

Result 3:

3(a). Every line of the type g_3 belonging to the set F_3 maps into a line g_3' in the $\ddot{\mathbf{x}}$ space (Fig. 6E), one end of which is the point S and the other end of which lies on

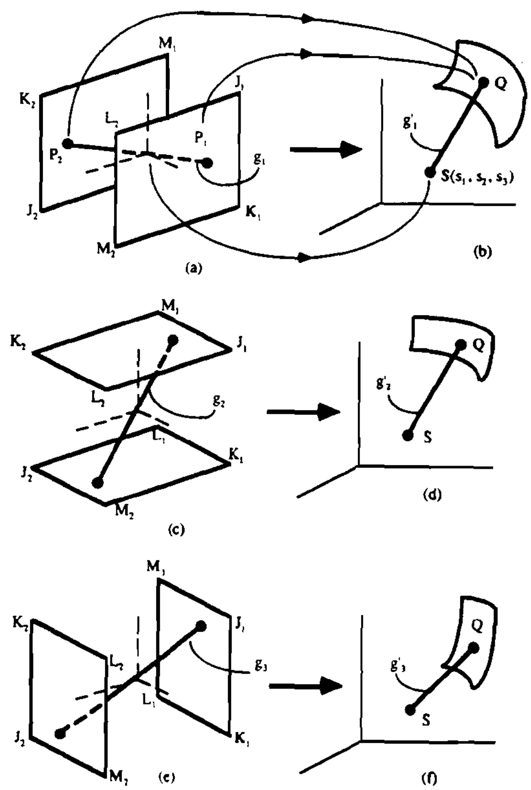

Fig. 6. Quadratic mappings for a three-degree-of-freedom spatial manipulator.

the quadratic surface patch (Fig. 6F) whose parametric equation (in $\dot{q}_1$ and $\dot{q}_2$) is:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_{30}^2 + 2n_{11}\dot{q}_1\dot{q}_2 \\ \quad + 2n_{12}\dot{q}_2\dot{q}_{30} + 2n_{13}\dot{q}_{30}\dot{q}_1 + s_1 \\ b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_{30}^2 + 2n_{21}\dot{q}_1\dot{q}_2 \\ \quad + 2n_{22}\dot{q}_2\dot{q}_{30} + 2n_{23}\dot{q}_{30}\dot{q}_1 + s_2 \\ b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_{30}^2 + 2n_{31}\dot{q}_1\dot{q}_2 \\ \quad + 2n_{32}\dot{q}_2\dot{q}_{30} + 2n_{33}\dot{q}_{30}\dot{q}_1 + s_3 \end{bmatrix}. \quad (61)$$

where

$$|\dot{q}_1| < \dot{q}_{10}, \quad |\dot{q}_2| < \dot{q}_{20}.$$

3(b). The set F_3 maps into a set $(S_{\dot{\mathbf{q}}})_3$ in the $\ddot{\mathbf{x}}$ plane that is a doubly infinite system of line segments, one end point of which is the point S and the other end point of which lies on the quadratic surface patch described by (61).

Result 4: The image set $(S_{\dot{\mathbf{q}}})$ of the joint velocity set F is the union of the sets $(S_{\dot{\mathbf{q}}})_1$, $(S_{\dot{\mathbf{q}}})_2$, $(S_{\dot{\mathbf{q}}})_3$ described above.

Proof of Results 1, 2, and 3: We will first derive certain useful properties of the quadratic mapping defined by equation (34):

$$\ddot{\mathbf{x}}^p = \mathbf{B}\langle\dot{\mathbf{q}}\rangle^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}.$$

The above equation can be written in the expanded form

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \cdot \begin{bmatrix} b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 \\ \quad + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1 \\ b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 \\ \quad + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2 \\ b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 \\ \quad + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3 \end{bmatrix}. \quad (62)$$

Consider the (input) $\dot{\mathbf{q}}$ space. It is convenient to think of this space as being generated by the continuous doubly infinite set of lines (also called a line congruence) passing through the origin with parametric equations

$$\begin{cases} \dot{q}_1 = t \\ \dot{q}_2 = m_1 t; \quad -\infty < m_1 < \infty, \ -\infty < m_2 < \infty \\ \dot{q}_3 = m_2 t \end{cases} \quad (63)$$

Each value of m_1 and m_2 gives us a member of the line congruence, a typical member of which is the line l shown in Figure 7. The image l' in the $\ddot{\mathbf{x}}$ space of the line l is obtained by substituting (63) into (62) and is described by the following parametric equations:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} m_1't^2 + s_1 \\ m_2't^2 + s_2 \\ m_3't^2 + s_3 \end{bmatrix}, \quad (64)$$

where

$$m_1' = b_{11} + b_{12}m_1^2 + b_{13}m_2^2 + 2n_{11}m_1 + 2n_{12}m_1m_2 + 2n_{13}m_2$$
$$m_2' = b_{21} + b_{22}m_1^2 + b_{23}m_2^2 + 2n_{21}m_1 + 2n_{22}m_1m_2 + 2n_{23}m_2$$
$$m_3' = b_{31} + b_{32}m_1^2 + b_{33}m_2^2 + 2n_{31}m_1 + 2n_{32}m_1m_2 + 2n_{33}m_2$$

From equation (63) and (64), one can infer the following facts:

Fact 1. The image of $l(viz.\ l')$ is a straight line.

Fact 2. The origin of the $\dot{\mathbf{q}}$ space maps into the point S (s_1, s_2, s_3) of the $\ddot{\mathbf{x}}$ space.

Fact 3. Two points with coordinates $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ and $(-\dot{q}_1, -\dot{q}_2, -\dot{q}_3)$ map into the same point of the $\ddot{\mathbf{x}}$ space.

These results are shown graphically in Figure 7.

Fact 1 follows from the fact that (64) is the equation of a straight line in the parameter t^2. Fact 2 follows from the fact that the point $(0, 0, 0)$ in the $\dot{\mathbf{q}}$ space,

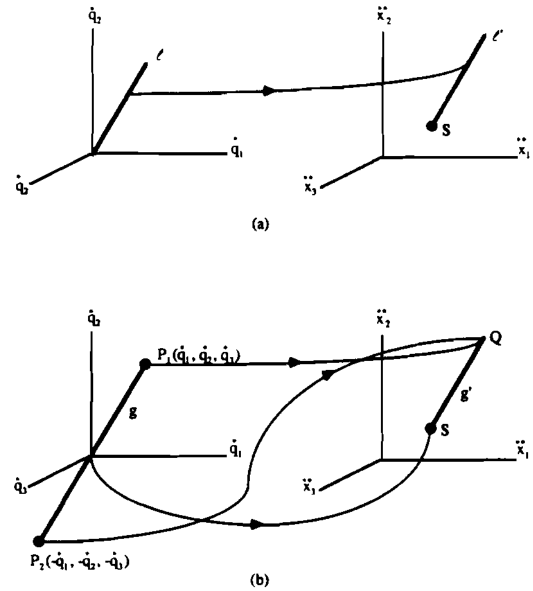

Fig. 7. Properties of the quadratic mapping for a three-degree-of-freedom spatial manipulator.

represented by the parameter $t = 0$ in (63), maps into the point (s_1, s_2, s_3) in the $\ddot{\mathbf{x}}$ space. If t is the parameter corresponding to the point $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ in the $\dot{\mathbf{q}}$ space, then, from (63), $-t$ is the parameter of the point $(-\dot{q}_1, -\dot{q}_2, -\dot{q}_3)$. From (64), we see that points with parameters t and $-t$ will map into the same point in the $\ddot{\mathbf{x}}$ space. This proves Fact 3.

The following two important properties of the quadratic mapping (33) (or (62)) follow directly from the above facts:

Property 1: The image of a line l passing through the origin of the $\dot{\mathbf{q}}$ space is the half-line l', one end point of which is the point $S(s_1, s_2, s_3)$ of the $\ddot{\mathbf{x}}$ space (see Fig. 7A).

Property 2: Consider a line segment g passing through the origin of the $\dot{\mathbf{q}}$ space and with endpoints $P_1(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ and $P_2(-\dot{q}_1, -\dot{q}_2, -\dot{q}_3)$ corresponding, respectively, to parameters t and $-t$; g maps into a line segment g' in the $\ddot{\mathbf{x}}$ plane, with one end point at $S(s_1, s_2, s_3)$ and the other end point at Q, whose coordinates are given by (64) (see Fig. 6B). Q is the image of points P_1 and P_2.

Property 1 is basically a statement of the fundamental "folding" property of the quadratic mapping. Property 2 is more useful for our purposes.

We now determine the image, under the mapping (34), of the set F_1 that consists of the doubly infinite system of line segments of the type g_1 (see Fig. 6A), which passes through the origin and has end points P_1 and P_2, respectively, on planes $J_1K_1M_2L_2$ and $M_1L_1J_2K_2$ (see Fig. 6A).

The plane $J_1K_1M_2L_2$ is described by

$$\dot{q}_1 = \dot{q}_{10}, \tag{65}$$

and the plane $M_1L_1J_2K_2$ is described by

$$\dot{q}_1 = -\dot{q}_{10}. \tag{66}$$

Therefore, if P_1 lying on $J_1K_1M_2L_2$ has coordinates $(\dot{q}_{10}, \dot{q}_2, \dot{q}_3)$, then P_2 lying on $M_1L_1J_2K_2$ has coordinates $(-\dot{q}_{10}, -\dot{q}_2, -\dot{q}_3)$. By property 2 of the quadratic mapping, the line segment g_1 with end points P_1 and P_2 will map into a line segment with one end point at $S(s_1, s_2, s_3)$ and the other end point at Q (Fig. 7), which is the image of both P_1 and P_2 and which we need to determine next. For every point $P_1(\dot{q}_{10}, \dot{q}_2, \dot{q}_3)$ lying in the plane $J_1K_1M_2L_2$, there is a point $P_2(-\dot{q}_{10}, -\dot{q}_2, -\dot{q}_3)$ lying in the plane $M_1L_1J_2K_2$, which, by Fact 3 established above, has the same image as P_1. Therefore, planes $J_1K_1M_2L_2$ and $M_1L_1J_2K_2$ have the same image. It is sufficient therefore to determine the image of plane $J_1K_1M_2L_2$. Because plane $J_1K_1M_2L_2$ is the set of all possible P_1, the image of $J_1K_1M_2L_2$ is the set of the images of all possible P_1. To obtain the image of $J_1K_1M_2L_2$, we substitute its equation (65) into (62) to obtain (59), which, because it is quadratic in the parameters $\dot{q}_1$ and $\dot{q}_2$, represents a quadratic surface patch in the $\ddot{\mathbf{x}}$ plane.

The quadratic surface patch (59) is the image of the plane $M_1L_1J_2K_2$, as well as the image of the plane $J_1K_1M_2L_2$. Any point P_1 of $M_1L_1J_2K_2$ with coordinates $(\dot{q}_{10}, \dot{q}_2, \dot{q}_3)$ and any point P_2 of $J_1K_1M_2L_2$ with coordinates $(-\dot{q}_{10}, -\dot{q}_2, -\dot{q}_3)$ will have the same image Q with coordinates $(\ddot{x}_1, \ddot{x}_2, \ddot{x}_3)$ given by (64).

We have thus shown that the line segment with the end points P_1 and P_2 will map into a line segment in the $\ddot{\mathbf{x}}$ plane with one end point at $S(s_1, s_2, s_3)$ and the other end point Q lying on the quadratic surface (59). This completes Result 1(a).

It is now a simple matter to determine the image $(S_{\dot{\mathbf{q}}})_1$ of F_1. By result 1(a), the doubly infinite set of line segments F_1 of the type g_1 with end points $P_1(\dot{q}_{10}, \dot{q}_2, \dot{q}_3)$ and $P_2(-\dot{q}_{10}, -\dot{q}_2, -\dot{q}_3)$ lying, respectively, in the planes $M_1L_1J_2K_2$ and $J_1K_1M_2L_2$ will map into the doubly infinite set of line segments $(S_{\dot{\mathbf{q}}})_1$ with one end point (always) at S and the other end point on the quadratic surface (59). This completes the proof of Result 1(b).

In exactly similar fashion, we can show Results 2(a) and 2(b) and Results 3(a) and 3(b).

Proof of Result 4: Because the images of F_1, F_2, and F_3 are, respectively, $(S_{\dot{\mathbf{q}}})_1, (S_{\dot{\mathbf{q}}})_2$, and $(S_{\dot{\mathbf{q}}})_3$, the image of $F = F_1 \cup F_2 \cup F_3$ is $S_{\dot{\mathbf{q}}} = (S_{\dot{\mathbf{q}}})_1 \cup (S_{\dot{\mathbf{q}}})_2 \cup (S_{\dot{\mathbf{q}}})_3$. $(S_{\dot{\mathbf{q}}})_1, (S_{\dot{\mathbf{q}}})_2$ and $(S_{\dot{\mathbf{q}}})_3$ have been defined, respectively, in Results 1(b), 2(b), and 3(b). This completes the proof of Result 2.

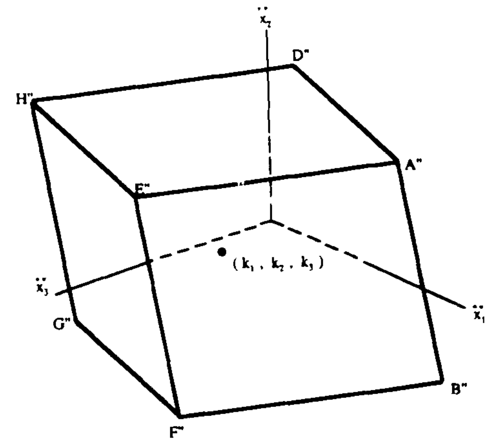

Fig. 8. The state acceleration $\mathbf{S_u}$ *for a three-degree-of-freedom spatial manipulator.*

Comment: The analytical description of $(S_{\dot{\mathbf{q}}})$ by means of $(S_{\dot{\mathbf{q}}})_1, (S_{\dot{\mathbf{q}}})_2$, and $(S_{\dot{\mathbf{q}}})_3$ is sufficient for the extraction of the acceleration properties of interest.

4.3. Determination of the State Acceleration Set $S_{\mathbf{u}}$

The state acceleration $S_{\mathbf{u}}$ corresponding to a state $\mathbf{u} = (\mathbf{q}, \dot{\mathbf{q}})^T$ of the spatial manipulator was defined by equation (41) and is the image set of the actuator torque set T under the mapping (40). We obtain the following results for the state acceleration set $S_{\mathbf{u}}$.

Result 1: For every element $\ddot{\mathbf{x}}(S_\tau)$ of the image set S_τ, there is a corresponding element $\ddot{\mathbf{x}}(S_{\mathbf{u}})$ of the state acceleration set $S_{\mathbf{u}}$, given by

$$\ddot{\mathbf{x}}(S_{\mathbf{u}}) = \ddot{\mathbf{x}}(S_\tau) + \mathbf{k}(\mathbf{q}, \dot{\mathbf{q}}), \tag{67}$$

where

$$\mathbf{k}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \mathbf{B}\langle\dot{\mathbf{q}}\rangle^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}. \tag{68}$$

Result 2: The state acceleration set $S_{\mathbf{u}}$, corresponding to a state $\mathbf{u} = (\mathbf{q}, \dot{\mathbf{q}})^T$ of the spatial three-degree-of-freedom manipulator is the parallelopiped $A''B''C''D''E''F''G''H''$ shown in Figure 8 and obtained by translating the set S_τ by the vector $\mathbf{k}(\mathbf{q}, \dot{\mathbf{q}})$ in the $\ddot{\mathbf{x}}$ space. The centroid of $S_{\mathbf{u}}$ is (k_1, k_2, k_3).

5. Properties of the Acceleration Sets

In this section, we explain how to characterize the image set, $S_\tau, S_{\dot{\mathbf{q}}}$, and the state acceleration set, $S_{\mathbf{u}}$, using the results in Section 4.

5.1. *Properties of the Acceleration Set* S_τ

We characterize the image set S_τ of the linear mapping as follows.

Result 1: The maximum acceleration of the acceleration set S_τ is denoted by $a_{max}(S_\tau)$ and is given by

$$a_{max}(S_\tau) = \max[d(OA'), d(OB'), d(OC'), d(OD')] \tag{69}$$

where

$$d(OA') = \sqrt{\begin{aligned}&(a_{11}\tau_{10} + a_{12}\tau_{20} + a_{13}\tau_{30})^2 \\ &\quad + (a_{21}\tau_{10} + a_{22}\tau_{20} + a_{23}\tau_{30})^2 \\ &\quad + (a_{31}\tau_{10} + a_{32}\tau_{20} + a_{33}\tau_{30})^2\end{aligned}}$$

$$d(OB') = \sqrt{\begin{aligned}&(a_{11}\tau_{10} - a_{12}\tau_{20} + a_{13}\tau_{30})^2 \\ &\quad + (a_{21}\tau_{10} - a_{22}\tau_{20} + a_{23}\tau_{30})^2 \\ &\quad + (a_{31}\tau_{10} - a_{32}\tau_{20} + a_{33}\tau_{30})^2\end{aligned}}$$

$$d(OC') = \sqrt{\begin{aligned}&(-a_{11}\tau_{10} - a_{12}\tau_{20} + a_{13}\tau_{30})^2 \\ &\quad + (-a_{21}\tau_{10} - a_{22}\tau_{20} + a_{23}\tau_{30})^2 \\ &\quad + (-a_{31}\tau_{10} - a_{32}\tau_{20} + a_{33}\tau_{30})^2\end{aligned}}$$

$$d(OD') = \sqrt{\begin{aligned}&(-a_{11}\tau_{10} + a_{12}\tau_{20} + a_{13}\tau_{30})^2 \\ &\quad + (-a_{21}\tau_{10} + a_{22}\tau_{20} + a_{23}\tau_{30})^2 \\ &\quad + (-a_{31}\tau_{10} + a_{32}\tau_{20} + a_{33}\tau_{30})^2\end{aligned}}$$

Result 2: The isotropic acceleration of the acceleration set S_τ is denoted by $a_{iso}(S_\tau)$ and is given by

$$a_{iso} = \min[\rho(A'B'F'E'), \rho(A'D'H'E'), \rho(A'B'C'D')] \tag{70}$$

where

$$\rho(A'B'F'E') = \frac{|\det(\mathbf{A})|\tau_{10}}{\sqrt{\begin{aligned}&(a_{22}a_{33} - a_{23}a_{32})^2 + (a_{12}a_{33} - a_{13}a_{32})^2 \\ &\quad + (a_{12}a_{23} - a_{13}a_{22})^2\end{aligned}}},$$

$$\rho(A'D'H'E') = \frac{|\det(\mathbf{A})|\tau_{20}}{\sqrt{\begin{aligned}&(a_{21}a_{33} - a_{23}a_{31})^2 + (a_{11}a_{33} - a_{13}a_{31})^2 \\ &\quad + (a_{11}a_{23} - a_{13}a_{21})^2\end{aligned}}},$$

$$\rho(A'B'C'D') = \frac{|\det(\mathbf{A})|\tau_{30}}{\sqrt{\begin{aligned}&(a_{21}a_{32} - a_{22}a_{31})^2 + (a_{11}a_{32} - a_{12}a_{31})^2 \\ &\quad + (a_{11}a_{22} - a_{12}a_{21})^2\end{aligned}}}.$$

5.2. *Properties of the Acceleration Set* $S_{\dot{q}}$

We characterize the image set $S_{\dot{q}}$ by the maximum acceleration $a_{max}(S_{\dot{q}})$ and by the maximum distance of any element of $S_{\dot{q}}$ from the reference planes P'_1, P'_2, and P'_3 defined by equations (56), (57), (58), respectively.

DEFINITION 1. Let $f_i, (i = 1, 2, 3)$, denote, respectively, the following cubic functions in the joint variable rates $\dot{q}_i, (i = 1, 2, 3)$:

$$\begin{aligned}
f_1(\dot{q}_1, \dot{q}_2, \dot{q}_3) = {} & (b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 \\
& + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1)(b_{11}\dot{q}_1 \\
& + n_{11}\dot{q}_2 + n_{13}\dot{q}_3) + (b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 \\
& + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 \\
& + 2n_{23}\dot{q}_3\dot{q}_1 + s_2)(b_{21}\dot{q}_1 + n_{21}\dot{q}_2 \\
& + n_{23}\dot{q}_3) + (b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 \\
& + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 \\
& + s_3)(b_{31}\dot{q}_1 + n_{31}\dot{q}_2 + n_{33}\dot{q}_3) = 0, \\
f_2(\dot{q}_1, \dot{q}_2, \dot{q}_3) = {} & (b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 \\
& + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1)(b_{12}\dot{q}_1 \\
& + n_{11}\dot{q}_2 + n_{13}\dot{q}_3) + (b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 \\
& + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 \\
& + 2n_{23}\dot{q}_3\dot{q}_1 + s_2)(b_{21}\dot{q}_1 + n_{21}\dot{q}_2 \\
& + n_{23}\dot{q}_3) + (b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 \\
& + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 \\
& + s_3)(b_{31}\dot{q}_1 + n_{31}\dot{q}_2 + n_{33}\dot{q}_3) = 0, \\
f_3 = {} & (b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 \\
& + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1)(b_{11}\dot{q}_1 \\
& + n_{11}\dot{q}_2 + n_{13}\dot{q}_3) + (b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 \\
& + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 \\
& + 2n_{23}\dot{q}_3\dot{q}_1 + s_2)(b_{21}\dot{q}_1 + n_{21}\dot{q}_2 \\
& + n_{23}\dot{q}_3) + (b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 \\
& + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 \\
& + s_3)(b_{31}\dot{q}_1 + n_{31}\dot{q}_2 + n_{33}\dot{q}_3) = 0.
\end{aligned} \tag{73}$$

where $f_i(\dot{q}_1, \dot{q}_2, \dot{q}_3)$, $(i = 1, 2, 3)$, is cubic in $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$.

DEFINITION 2. It is useful in our derivations to be able to refer to the solutions of certain equations that play an important role in obtaining the maximum acceleration of $S_{\dot{q}}$, $a_{max}(S_{\dot{q}})$. Each equation or equation pair of interest is given in column 1 of Table 1, and the corresponding variables are indicated in column 2. All equations in column 1 are cubics in the variables in column 2. The notation used to denote the solution of each equation or equation pair is given in column 3.

DEFINITION 3.

$$l(\dot{\mathbf{q}}) \triangleq l(\dot{q}_1, \dot{q}_2, \dot{q}_3)$$

$$\triangleq \begin{pmatrix} (b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 \\ \quad + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1)^2 \\ + (b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 \\ \quad + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2)^2 \\ + (b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 \\ \quad + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3)^2 \end{pmatrix}^{\frac{1}{2}}. \tag{74}$$

Table 1. Solutions of Cubic Equations

Equations	Variables	Notation Used to Denote Solutions
$f_2(\dot{q}_{10}, \dot{q}_2, \dot{q}_3) = 0$ and $f_3(\dot{q}_{10}, \dot{q}_2, \dot{q}_3) = 0$	$\dot{q}_2, \dot{q}_3$	$\dot{q}_2^{(1)}, \dot{q}_3^{(1)}$
$f_3(\dot{q}_1, \dot{q}_{20}, \dot{q}_3) = 0$ and $f_1(\dot{q}_1, \dot{q}_{20}, \dot{q}_3) = 0$	$\dot{q}_3, \dot{q}_1$	$\dot{q}_3^{(2)}, \dot{q}_1^{(2)}$
$f_1(\dot{q}_1, \dot{q}_2, \dot{q}_{30}) = 0$ and $f_2(\dot{q}_1, \dot{q}_2, \dot{q}_{30}) = 0$	$\dot{q}_1, \dot{q}_1$	$\dot{q}_1^{(3)}, \dot{q}_2^{(3)}$
$f_1(\dot{q}_1, \dot{q}_{20}, \dot{q}_{30}) = 0$	$\dot{q}_1$	$\dot{q}_1^{(4)}$
$f_1(\dot{q}_1, -\dot{q}_{20}, \dot{q}_{30}) = 0$	$\dot{q}_1$	$\dot{q}_1^{(5)}$
$f_2(\dot{q}_{10}, \dot{q}_2, \dot{q}_{30}) = 0$	$\dot{q}_2$	$\dot{q}_2^{(6)}$
$f_2(\dot{q}_{10}, \dot{q}_2, -\dot{q}_{30}) = 0$	$\dot{q}_2$	$\dot{q}_2^{(7)}$
$f_3(\dot{q}_{10}, \dot{q}_{20}, -\dot{q}_3) = 0$	$\dot{q}_3$	$\dot{q}_3^{(8)}$
$f_3(\dot{q}_{10}, -\dot{q}_{20}, \dot{q}_3) = 0$	$\dot{q}_3$	$\dot{q}_3^{(9)}$

DEFINITION 4. Let h_i, $(i = 1, 2, 3)$ denote, respectively, the following linear equations in the joint variable rates, $\dot{q}_i$, $i = 1, 2, 3$:

$$h_1(\dot{q}_1, \dot{q}_2, \dot{q}_3) = \left[\begin{array}{l} (a_{22}a_{33} - a_{23}a_{32})(b_{11}\dot{q}_1 + n_{11}\dot{q}_2 \\ \quad + n_{13}\dot{q}_3) + (a_{13}a_{32} - a_{12}a_{33})(b_{21}\dot{q}_1 \\ \quad + n_{21}\dot{q}_2 + n_{23}\dot{q}_3) + (a_{12}a_{23} \\ \quad - a_{13}a_{22})(b_{31}\dot{q}_1 + n_{31}\dot{q}_2 + n_{33}\dot{q}_3) \end{array}\right] \tag{75}$$

$$h_2(\dot{q}_1, \dot{q}_2, \dot{q}_3) = \left[\begin{array}{l} (a_{22}a_{33} - a_{23}a_{32})(b_{12}\dot{q}_1 + n_{11}\dot{q}_2 \\ \quad + n_{12}\dot{q}_3) + (a_{13}a_{32} - a_{12}a_{33})(b_{22}\dot{q}_1 \\ \quad + n_{21}\dot{q}_2 + n_{23}\dot{q}_3) + (a_{12}a_{23} \\ \quad - a_{13}a_{22})(b_{32}\dot{q}_1 + n_{31}\dot{q}_2 + n_{32}\dot{q}_3) \end{array}\right] \tag{76}$$

$$h_3(\dot{q}_1, \dot{q}_2, \dot{q}_3) = \left[\begin{array}{l} (a_{22}a_{33} - a_{23}a_{32})(b_{13}\dot{q}_1 + n_{12}\dot{q}_2 \\ \quad + n_{13}\dot{q}_3) + (a_{13}a_{32} - a_{12}a_{33})(b_{23}\dot{q}_1 \\ \quad + n_{22}\dot{q}_2 + n_{23}\dot{q}_3) + (a_{12}a_{23} \\ \quad - a_{13}a_{22})(b_{33}\dot{q}_1 + n_{32}\dot{q}_2 + n_{33}\dot{q}_3) \end{array}\right] \tag{77}$$

where $h_i(\dot{q}_1, \dot{q}_2, \dot{q}_3), (i = 1, 2, 3)$ is linear in $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$.

DEFINITION 5. It is also useful to be able to refer to the solutions of certain equations that play an important role in obtaining $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_i)$, $i = 1, 2, 3$, defined below. In Table 2, each equation or equation pair of interest is given in column 1, and the corresponding variables are indicated in column 2. All equations in column 1 are linear in the variables in column 2. The notation used to denote the solution of each equation or equation pair is given in column 3.

DEFINITION 6.

$$\sigma_1(\dot{q}_1, \dot{q}_2, \dot{q}_3) = \left[\begin{array}{c} (a_{22}a_{33} - a_{23}a_{32})^2 + (a_{12}a_{33} - a_{32}a_{13})^2 \\ + (a_{12}a_{23} - a_{13}a_{22})^2 \end{array}\right]^{-\frac{1}{2}} \times \left(\begin{array}{l} (a_{22}a_{33} - a_{23}a_{32})2(b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 \\ \quad + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 \\ \quad + 2n_{13}\dot{q}_3\dot{q}_1 + s_1) + (a_{32}a_{13} - a_{12}a_{33}) \\ \quad \times 2(b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 \\ \quad + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2) \\ \quad + (a_{12}a_{23} - a_{13}a_{22})2(b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 \\ \quad + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 \\ \quad + 2n_{33}\dot{q}_3\dot{q}_1 + s_3) \end{array}\right), \tag{78}$$

$$\sigma_2(\dot{q}_1, \dot{q}_2, \dot{q}_3) = \left[\begin{array}{c} (a_{23}a_{31} - a_{13}a_{33})^2 + (a_{13}a_{31} - a_{11}a_{33})^2 \\ + (a_{21}a_{13} - a_{11}a_{23})^2 \end{array}\right]^{-\frac{1}{2}} \times \left(\begin{array}{l} (a_{21}a_{33} - a_{23}a_{31})2(b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 \\ \quad + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 \\ \quad + 2n_{13}\dot{q}_3\dot{q}_1 + s_1) + (a_{31}a_{13} - a_{11}a_{33}) \\ \quad \times 2(b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 \\ \quad + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2) \\ \quad + (a_{11}a_{23} - a_{13}a_{21})2(b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 \\ \quad + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 \\ \quad + 2n_{33}\dot{q}_3\dot{q}_1 + s_3) \end{array}\right) \tag{79}$$

$$\sigma_3(\dot{q}_1, \dot{q}_2, \dot{q}_3) = \left[\begin{array}{c} (a_{32}a_{21} - a_{22}a_{31})^2 + (a_{11}a_{32} - a_{31}a_{12})^2 \\ + (a_{11}a_{22} - a_{12}a_{21})^2 \end{array}\right]^{-\frac{1}{2}} \times \left(\begin{array}{l} (a_{32}a_{21} - a_{22}a_{31})2(b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 \\ \quad + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 \\ \quad + 2n_{13}\dot{q}_3\dot{q}_1 + s_1) + (a_{31}a_{12} - a_{11}a_{32}) \\ \quad \times 2(b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 \\ \quad + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2) \\ \quad + (a_{11}a_{22} - a_{12}a_{21})2(b_{31}\dot{q}_1^2 \\ \quad + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 \\ \quad + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3) \end{array}\right) \tag{80}$$

DEFINITION 7. Let $\rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1)$, $\rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_2)$ and $\rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_3)$ denote, respectively, the distance of any point $\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}})$ of $S_{\dot{\mathbf{q}}}$ from the planes p_1', p_2', and p_3' defined in Section 4.1. We define the following:

$$\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1') \triangleq \max \rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1'), \tag{81}$$

$$\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_2') \triangleq \max \rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_2'), \tag{82}$$

$$\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_3') \triangleq \max \rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_3'). \tag{83}$$

The quantity $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1')$, for example, represents the distance of that point of $S_{\dot{\mathbf{q}}}$ that is farthest from plane p_1'; $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1')$, $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_2')$ and $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_3')$ are necessary for determining the local isotropic acceleration in Section 5.4.

Result 1: For a general three-degree-of-freedom spatial manipulator, the maximum acceleration of the acceleration set $S_{\dot{\mathbf{q}}}$ will be denoted by $a_{\max}(S_{\dot{\mathbf{q}}})$ and is given

Table 2. Solutions of Linear Equations

Equations	Variables	Notation Used to Denote Solutions
$h_2(\dot q_{10}, \dot q_2, \dot q_3) = 0$ and $h_3(\dot q_{10}, \dot q_2, \dot q_3) = 0$	$\dot q_2, \dot q_3$	$\dot q_2^{[1]}, \dot q_3^{[1]}$
$h_3(\dot q_1, \dot q_{20}, \dot q_3) = 0$ and $h_1(\dot q_1, \dot q_{20}, \dot q_3) = 0$	$\dot q_3, \dot q_1$	$\dot q_3^{[2]}, \dot q_1^{[2]}$
$h_1(\dot q_1, \dot q_2, \dot q_{30}) = 0$ and $h_2(\dot q_1, \dot q_2, \dot q_{30}) = 0$	$\dot q_1, \dot q_2$	$\dot q_1^{[3]}, \dot q_2^{[3]}$
$h_1(\dot q_1, \dot q_{20}, \dot q_{30}) = 0$	$\dot q_1$	$\dot q_1^{[4]}$
$h_1(\dot q_1, -\dot q_{20}, \dot q_{30}) = 0$	$\dot q_1$	$\dot q_1^{[5]}$
$h_2(\dot q_{10}, \dot q_2, \dot q_{30}) = 0$	$\dot q_2$	$\dot q_2^{[6]}$
$h_2(\dot q_{10}, \dot q_2, -\dot q_{30}) = 0$	$\dot q_2$	$\dot q_2^{[7]}$
$h_3(\dot q_{10}, \dot q_{20}, -\dot q_3) = 0$	$\dot q_3$	$\dot q_3^{[8]}$
$h_3(\dot q_{10}, -\dot q_{20}, \dot q_3) = 0$	$\dot q_3$	$\dot q_3^{[9]}$

by

$$a_{\max}(S_{\dot{\mathbf q}}) = \max[l_{(1)}, l_{(2)}, \ldots, l_{(13)}], \tag{84}$$

where

$$\begin{aligned}
l_{(1)} &= l(\dot q_{10}, \dot q_2^{(1)}, \dot q_3^{(1)}),\\
l_{(2)} &= l(\dot q_1^{(2)}, \dot q_{20}, \dot q_3^{(2)}),\\
l_{(3)} &= l(\dot q_1^{(3)}, \dot q_2^{(3)}, \dot q_{30}),\\
l_{(4)} &= l(\dot q_1^{(4)}, \dot q_{20}, \dot q_{30}),\\
l_{(5)} &= l(\dot q_1^{(5)}, -\dot q_{20}, \dot q_{30}),\\
l_{(6)} &= l(\dot q_{10}, \dot q_2^{(6)}, \dot q_{30}),\\
l_{(7)} &= l(\dot q_{10}, \dot q_2^{(7)}, -\dot q_{30}),\\
l_{(8)} &= l(\dot q_{10}, \dot q_{20}, \dot q_3^{(8)}),\\
l_{(9)} &= l(\dot q_{10}, -\dot q_{20}, \dot q_3^{(9)}),\\
l_{(10)} &= l(\dot q_{10}, \dot q_{20}, \dot q_{30}),\\
l_{(11)} &= l(\dot q_{10}, \dot q_{20}, -\dot q_{30}),\\
l_{(12)} &= l(\dot q_{10}, -\dot q_{20}, -\dot q_{30}),\\
l_{(13)} &= l(\dot q_{10}, -\dot q_{20}, \dot q_{30}).
\end{aligned}$$

Result 2: For a general three-degree-of-freedom manipulator, the maximum distance from an element of $S_{\dot{\mathbf q}}$ to the reference planes p_1', p_2', and p_3' are given, respectively, by

$$\begin{aligned}
&\max[\rho(\ddot{\mathbf x}(S_{\dot{\mathbf q}}), p_i')], i = 1,2,3 \qquad (85)\\
&= \max[(\sigma_i)_{[1]}, (\sigma_i)_{[2]}, \ldots, (\sigma_i)_{[13]}],\\
&(i = 1,2,3).
\end{aligned}$$

where

$$\begin{aligned}
(\sigma_i)_{[1]} &= \sigma_i(\dot q_{10}, \dot q_2^{[1]}, \dot q_3^{[1]}),\\
(\sigma_i)_{[2]} &= \sigma_i(\dot q_1^{[2]}, \dot q_{20}, \dot q_3^{[2]}),\\
(\sigma_i)_{[3]} &= \sigma_i(\dot q_1^{[3]}, \dot q_2^{[3]}, \dot q_{30}),\\
(\sigma_i)_{[4]} &= \sigma_i(\dot q_1^{[4]}, \dot q_{20}, \dot q_{30}),\\
(\sigma_i)_{[5]} &= \sigma_i(\dot q_1^{[5]}, -\dot q_{20}, \dot q_{30}),\\
(\sigma_i)_{[6]} &= \sigma_i(\dot q_{10}, \dot q_2^{[6]}, \dot q_{30}),\\
(\sigma_i)_{[7]} &= \sigma_i(\dot q_{10}, \dot q_2^{[7]}, -\dot q_{30}),\\
(\sigma_i)_{[8]} &= \sigma_i(\dot q_{10}, \dot q_{20}, \dot q_3^{[8]}),\\
(\sigma_i)_{[9]} &= \sigma_i(\dot q_{10}, -\dot q_{20}, \dot q_3^{[9]}),\\
(\sigma_i)_{[10]} &= \sigma_i(\dot q_{10}, \dot q_{20}, \dot q_{30}),\\
(\sigma_i)_{[11]} &= \sigma_i(\dot q_{10}, \dot q_{20}, -\dot q_{30}),\\
(\sigma_i)_{[12]} &= \sigma_i(\dot q_{10}, -\dot q_{20}, -\dot q_{30}),\\
(\sigma_i)_{[13]} &= \sigma_i(\dot q_{10}, -\dot q_{20}, \dot q_{30}),
\end{aligned}$$

where $\sigma_i(\dot q_1, \dot q_2, \dot q_3)$, $(i = 1,2,3)$, are defined by equations (78), (79), and (80).

5.3. Properties of the State Acceleration Set

DEFINITION:

K: centroid of the acceleration set in the $\ddot{\mathbf x}$ space with coordinates (k_1, k_2, k_3) given by (39).

$\rho(K, p_1)$: distance from point K to the reference plane p_1'.

$\rho(K, p_2)$: distance from point K to the reference plane p_2'.

$\rho(K, p_3)$: distance from point K to the reference plane p_3'.

$\rho(A'B'F'E'), \rho(A''B''F''E''), \ldots$: distance from the origin to plane $A'B'F'E', A''B''F''E'', \ldots$

Result 1: The maximum acceleration corresponding to any dynamic state $\mathbf u$ of the manipulator is denoted by $a_{\max}(S_{\mathbf u})$ and is given by

$$\begin{aligned}
a_{\max}(S_{\mathbf u}) = \max\big[&d(OA''), d(OB''), d(OC''), d(OD''),\\
&d(OE''), d(OF''), d(OG''), d(OH'')\big]
\end{aligned} \tag{86}$$

where

$$d(OA'') = \sqrt{\begin{aligned}&(a_{11}\tau_{10} + a_{12}\tau_{20} + a_{13}\tau_{30} + k_1)^2 \\ &+ (a_{21}\tau_{10} + a_{22}\tau_{20} + a_{23}\tau_{30} + k_2)^2 \\ &+ (a_{31}\tau_{10} + a_{32}\tau_{20} + a_{33}\tau_{30} + k_3)^2\end{aligned}}$$

$$d(OB'') = \sqrt{\begin{aligned}&(a_{11}\tau_{10} - a_{12}\tau_{20} + a_{13}\tau_{30} + k_1)^2 \\ &+ (a_{21}\tau_{10} - a_{22}\tau_{20} + a_{23}\tau_{30} + k_2)^2 \\ &+ (a_{31}\tau_{10} - a_{32}\tau_{20} + a_{33}\tau_{30} + k_3)^2\end{aligned}}$$

$$d(OC'') = \sqrt{\begin{aligned}&(a_{11}\tau_{10} + a_{12}\tau_{20} - a_{13}\tau_{30} - k_1)^2 \\ &+ (a_{21}\tau_{10} + a_{22}\tau_{20} - a_{23}\tau_{30} - k_2)^2 \\ &+ (a_{31}\tau_{10} + a_{32}\tau_{20} - a_{33}\tau_{30} - k_3)^2\end{aligned}}$$

$$d(OD'') = \sqrt{\begin{aligned}&(a_{11}\tau_{10} - a_{12}\tau_{20} - a_{13}\tau_{30} - k_1)^2 \\ &+ (a_{21}\tau_{10} - a_{22}\tau_{20} - a_{23}\tau_{30} - k_2)^2 \\ &+ (a_{31}\tau_{10} - a_{32}\tau_{20} - a_{33}\tau_{30} - k_3)^2\end{aligned}}$$

$$d(OE'') = \sqrt{\begin{aligned}&(a_{11}\tau_{10} + a_{12}\tau_{20} - a_{13}\tau_{30} + k_1)^2 \\ &+ (a_{21}\tau_{10} + a_{22}\tau_{20} - a_{23}\tau_{30} + k_2)^2 \\ &+ (a_{31}\tau_{10} + a_{32}\tau_{20} - a_{33}\tau_{30} + k_3)^2\end{aligned}}$$

$$d(OF'') = \sqrt{\begin{aligned}&(a_{11}\tau_{10} - a_{12}\tau_{20} - a_{13}\tau_{30} + k_1)^2 \\ &+ (a_{21}\tau_{10} - a_{22}\tau_{20} - a_{23}\tau_{30} + k_2)^2 \\ &+ (a_{31}\tau_{10} - a_{32}\tau_{20} - a_{33}\tau_{30} + k_3)^2\end{aligned}}$$

$$d(OG'') = \sqrt{\begin{aligned}&(a_{11}\tau_{10} + a_{12}\tau_{20} + a_{13}\tau_{30} - k_1)^2 \\ &+ (a_{21}\tau_{10} + a_{22}\tau_{20} + a_{23}\tau_{30} - k_2)^2 \\ &+ (a_{31}\tau_{10} + a_{32}\tau_{20} + a_{33}\tau_{30} - k_3)^2\end{aligned}}$$

$$d(OH'') = \sqrt{\begin{aligned}&(a_{11}\tau_{10} - a_{12}\tau_{20} + a_{13}\tau_{30} - k_1)^2 \\ &+ (a_{21}\tau_{10} - a_{22}\tau_{20} + a_{23}\tau_{30} - k_2)^2 \\ &+ (a_{31}\tau_{10} - a_{32}\tau_{20} + a_{33}\tau_{30} - k_3)^2\end{aligned}}$$

Result 2: The necessary and sufficient conditions for the existence of the isotropic acceleration are the following:

$$|\det(\mathbf{A})|\tau_{10} - |(a_{22}a_{33} - a_{23}a_{32})k_1 + (a_{13}a_{32} - a_{12}a_{33})k_2 + (a_{12}a_{23} - a_{22}a_{13})k_3| > 0, \quad (87)$$

$$|\det(\mathbf{A})|\tau_{20} - |(a_{21}a_{33} - a_{23}a_{31})k_1 + (a_{11}a_{33} - a_{13}a_{31})k_2 + (a_{11}a_{23} - a_{21}a_{13})k_3| > 0, \quad (88)$$

$$|\det(\mathbf{A})|\tau_{30} - |(a_{21}a_{32} - a_{22}a_{31})k_1 + (a_{12}a_{31} - a_{11}a_{32})k_2 + (a_{11}a_{22} - a_{12}a_{21})k_3| > 0, \quad (89)$$

Result 3: The isotropic acceleration corresponding to any dynamic state $\mathbf{u}$ of the manipulator is denoted by $a_{iso}(S_{\mathbf{u}})$ and is given by

$$\min \begin{bmatrix} \dfrac{|\det(\mathbf{A})|\tau_{10} - |(a_{22}a_{31} - a_{23}a_{31})k_1 + (a_{13}a_{32} - a_{12}a_{33})k_2 + (a_{22}a_{23} - a_{22}a_{13})k_3|}{\sqrt{(a_{22}a_{31} - a_{23}a_{31})^2 + (a_{13}a_{32} - a_{12}a_{33})^2 + (a_{12}a_{23} - a_{22}a_{13})^2}}, \\ \dfrac{|\det(\mathbf{A})|\tau_{20} - |(a_{23}a_{31} - a_{21}a_{33})k_1 + (a_{11}a_{33} - a_{13}a_{31})k_2 + (a_{11}a_{23} - a_{21}a_{13})k_3|}{\sqrt{(a_{23}a_{31} - a_{21}a_{33})^2 + (a_{11}a_{33} - a_{13}a_{31})^2 + (a_{11}a_{23} - a_{21}a_{13})^2}}, \\ \dfrac{|\det(\mathbf{A})|\tau_{30} - |(a_{21}a_{32} - a_{22}a_{31})k_1 + (a_{12}a_{31} - a_{11}a_{32})k_2 + (a_{11}a_{22} - a_{12}a_{21})k_3|}{\sqrt{(a_{12}a_{32} - a_{22}a_{31})^2 + (a_{12}a_{31} - a_{11}a_{32})^2 + (a_{11}a_{22} - a_{12}a_{21})^2}} \end{bmatrix}. \quad (90)$$

5.4. Local Acceleration Properties

At any given (local) configuration $\mathbf{q}$ in the workspace, the following properties are of theoretical and practical importance:

- The maximum magnitude of the acceleration at any configuration $\mathbf{q}$ in the workspace
- The maximum magnitude of the isotropic acceleration at any configuration $\mathbf{q}$ in the workspace

Result 1: The local maximum acceleration $a_{max.local}$ of a spatial three-degree-of-freedom manipulator at a given configuration $\mathbf{q}$ is specified by

$$(a_{max.local})_{lb} \leq a_{max.local} \leq (a_{max.local})_{ub} \quad (91)$$

where $(a_{max.local})_{lb}$ is given by (86) with $k_1(\mathbf{q}, \dot{\mathbf{q}})$, $k_2(\mathbf{q}, \dot{\mathbf{q}})$, and $k_3(\mathbf{q}, \dot{\mathbf{q}})$ in equation (86) evaluated at that joint velocity vector $\dot{\mathbf{q}}$ that maximizes $l(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ in equation (84), and

$$(a_{max.local})_{ub} = a_{max}(S_{\dot{\mathbf{q}}}) + a_{max}(S_{\tau}) \quad (92)$$

where $a_{max}(S_{\dot{\mathbf{q}}})$ is given by (84) and $a_{max}(S_{\tau})$ is given by (69).

Result 2: The local isotropic acceleration $a_{iso.local}$ at a given configuration $\mathbf{q}$ is specified by

$$a_{iso.local}(S_{\mathbf{u}}) = \min \begin{bmatrix} \rho(A'B'F'E') - \rho_{max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1) \\ \rho(A'D'H'E') - \rho_{max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_2) \\ \rho(A'B'C'D') - \rho_{max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_3) \end{bmatrix} \quad (93)$$

where $\rho(A'B'F'E')$, $\rho(A'D'H'E')$, and $\rho(A'B'C'D')$ are given in equation (70) and where $\rho_{max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1)$, $\rho_{max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_2)$, and $\rho_{max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_3)$ are given by equation (85).

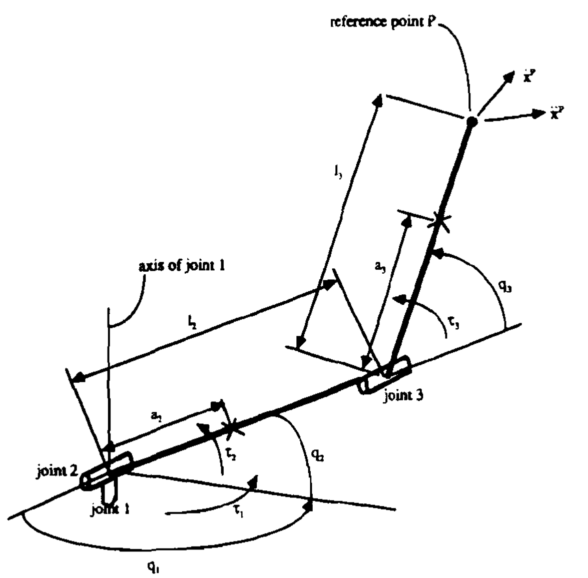

Fig. 9. Schematic diagram of a three-degree-of-freedom spatial manipulator.

Table 3. Parameters for the Spatial Manipulator*

$\ell_1 = 0.0$	$\ell_2 = 0.303$	$\ell_3 = 0.254$	(m)
$a_1 = 0.0$	$a_2 = 0.196$	$a_3 = 0.094$	(m)
$m_1 = 3.5$	$m_2 = 2.259$	$m_3 = 1.129$	(kg)
$I_1 = 1.2$	$J_1 = —$	$K_1 = —$	(kg-m^2)
$I_2 = .129$	$J_2 = .129$	$K_2 \cong 0$	(kg-m^2)
$I_3 = .003$	$J_3 = .003$	$K_3 \cong 0$	(kg-m^2)

*See Figure 9 and the Appendix.

6. Example

To demonstrate the ease of applicability of the general acceleration set theory for spatial manipulators developed in the previous sections, we have written simple computer codes to generate the acceleration properties of the various acceleration sets for a common type of three-degree-of-freedom spatial manipulator that is shown in Figure 9 and whose kinematical and dynamical equations are given in Kim (1989). (The axis of joint 1 in Figure 9 is vertical.) The actual geometric and inertia parameters used in the example are given in Table 3. The dynamical equations have been derived using Kane's dynamical equations (Kane and Levinson 1983, 1985; Desa and Roth 1985).

The configuration chosen was $q_1 = 0, q_2 = 45°$, and $q_3 = 45°$. The joint velocity constraints are

$$\dot{q}_i \leq \dot{q}_{i0} = 1 \text{rad/s}; \quad i = 1, 2, 3.$$

The torque constraints are

$$\tau_i \leq \tau_{i0}, \quad i = 1, 2, 3.$$

τ_{i0} may be thought of as the size (or maximum torque rating) of the actuators; the numerical values of τ_{10}, $(i = 1, 2, 3)$, are given in Table 4.

The properties of the state acceleration set were computed at $q_1 = 0$, $q_2 = 45$, and $q_3 = 45°$; $\dot{q}_1 = 1$ rad/s, $\dot{q}_3 = -1$ rad/s.

To show how the theory might be used for design purposes, we have determined the acceleration properties for three cases (see Table 4). Five acceleration properties have been determined in each case: the maximum and isotropic acceleration of the set $\mathbf{S}_\tau$, the maximum and isotropic acceleration of the state acceleration set, and the (local) isotropic acceleration at the configuration $(0, 45°, 45°)^T$.

In all three cases, the sizes of the first two actuators remain constant ($\tau_{10} = 35$ N-m and $\tau_{20} = 8.2$ N-m) and the size of the third actuator (driving link 3) is varied. In Case 1 of Table 4 ($\tau_{30} = 0.17$ N-m), the end effector does not have either a state or local isotropic acceleration. When the size of actuator 3 is increased to 0.4 N-m (Case 2), we obtain a state isotropic acceleration of 0.93 m/s^2, but the local isotropic acceleration is very small 0.03 m/s^2. Therefore, for given τ_{10} and τ_{20}, τ_{30} must be greater than 0.4 N-m in order to have a local isotropic acceleration at the specified configuration $\mathbf{q}$. Case 3 shows that for actuator size τ_{30} of 0.6 N-m, we have a local isotropic acceleration of 1.61 m/s^2. The designer must then decide (from past experience) whether this magnitude of isotropic acceleration is reasonable.

Comments

1. These computations can be repeated for various configurations in the workspace, after which decisions can be made regarding actuator sizes.
2. Algorithms for the determination of minimum actuator sizes to achieve a desired isotropic acceleration are given in (Desa and Kim 1990b) for the planar case. The extension to the spatial case is relatively straightforward.

7. Summary and Conclusions

In this article, we extended the acceleration set theory for planar manipulators, developed in Desa and Kim (1990a), to spatial manipulators. As in the planar case, we have accomplished the following:

- Given the kinematical and dynamical equations of a manipulator, we have defined the image set $\mathbf{S}_\tau$ corresponding to the set T of actuator torques and the image set $\mathbf{S}_{\dot{q}}$ corresponding to the set F of the joint velocities. We have also defined the state acceleration set $\mathbf{S}_\mathbf{u}$ at a specified point $\mathbf{u}$ in the state space.

Table 4. Acceleration Properties for the Manipulator of Section 6

	Actuator Torques			Acceleration		Properties		
Case	τ_{10} (N-m)	τ_{20} (N-m)	τ_{30} (N-m)	$a_{max}(S_\tau)$ (m/s^2)	$a_{iso}(S_\tau)$ (m/s^2)	$a_{max}(S_u)$ (m/s^2)	$a_{iso}(S_u)$ (m/s^2)	$a_{iso,\,local}$ (m/s^2)
1	35	8.2	0.17	20.3	1.35	23.7	0	0
2	35	8.2	0.4	25.06	3.16	29.1	0.93	0.03
3	35	8.2	0.6	30.3	4.75	33.9	2.51	1.61

- We have determined the image sets, S_τ and $S_{\dot{q}}$, and the state acceleration set S_u.
- We have characterized the image sets S_τ and the state acceleration set S_u by their maximum and isotropic acceleration. The image set $S_{\dot{q}}$ has been also characterized by its maximum acceleration.
- At a configuration or position, **q**, in the workspace, we have established two local acceleration properties: the local maximum acceleration and the local isotropic acceleration. The local maximum acceleration specifies the maximum magnitude of the acceleration of a reference point on the end effector. The local isotropic acceleration specifies the maximum magnitude of the available acceleration of the end effector in all directions.

We then demonstrated the application of the acceleration set theory for spatial manipulator to the three-degree-of-freedom spatial manipulator shown in Figure 9.

Therefore, we have demonstrated the hypothesis stated in the introduction—i.e., that the analytic properties of acceleration sets can be determined from the properties of the linear and quadratic mappings that define them (the acceleration sets). Furthermore, the acceleration properties of interest—especially the isotropic acceleration—have been determined in terms of the manipulator parameters, the torque limits, and the joint variable rate (joint velocity) limits. These results can therefore be applied to manipulator design problems as demonstrated in Desa and Kim (1990b).

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References

Craig, J. 1986. *Introduction to Robotics: Mechanics and Control*. Reading, MA: Addison-Wesley.

Desa, S., and Kim, Y. 1990a. The definition, determination, and characterization of acceleration sets for planar manipulators. *21st Biennial ASME Mechanisms Conference*, Chicago, pp. 207–215.

Desa, S., and Kim, Y. 1990b. The application of acceleration set theory to manipulator design. *21st Biennial ASME Mechanisms Conference*, Chicago, pp. 221–228.

Desa S., and Roth, B. 1985. Mechanics: Kinematics and Dynamics. In Beni, G., and Hackwood, S. (eds.): *Recent Advances in Robotics*. New York: John Wiley and Sons.

Graettinger, T., and Krogh, B. 1988. The acceleration radius: A global performance measure for robotic manipulators. *IEEE J. Robot. Automation* 4(1):60–69.

Kane, T. R., and Levinson, D. A. 1983. The use of Kane's dynamical equations in robotics. *Int. J. Robot. Res.* 2(3):3–21.

Kane, T. R., and Levinson, D. A. 1985. *Dynamics: Theory and Application*. New York: McGraw-Hill.

Khatib, O., and Burdick, J. 1987. Optimization of dynamics in manipulator design: The operational space formulation. *Int. J. Robot. Automation* 2(2):90–97.

Kim, Y. 1989. Acceleration sets of manipulators: Theory and application. Ph.D. thesis, Department of Mechanical Engineering, Carnegie Mellon University.

Semple, J. G., and Kneebone, G. T. 1952. *Algebraic Projective Geometry*. Oxford, England: Oxford University Press.

Yoshikawa, T. 1985. Dynamic manipulability of robot manipulators. *Proceedings of IEEE International Conference on Robotics and Automation*, pp. 1033–1038.