Selected Transl. in Math.
Statist. and Probability
Volume 10, 1971

ON THE PROBABILITY FUNCTIONAL OF DIFFUSION PROCESSES*

R. L. STRATONOVIČ

It is known that the multivariate probability density of a Markov diffusion process $x(t)$, $0 \leqslant t \leqslant T$, described by the equation

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[a(x,t)p\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[b(x,t)p\right] \tag{1}$$

can be written approximately in the form

$$p(x_0, x_1, \ldots, x_T) = p_0(x_0)\prod_{i=0}^{N-1}\left[2\pi b(x_i, t_i)\right]^{-\frac{1}{2}}$$

$$\times \exp\left\{-\frac{1}{2}\sum_{i=0}^{N-1}\left[\frac{x_{i+1}-x_i}{\Delta_i} - a(x_i, t_i)\right]^2 \frac{\Delta_i}{b(x_i, t_i)}\right\}, \tag{2}$$

where $x_i = x(t_i)$; $t_{i+1} - t_i = \Delta_i > 0$; $t_N = T$, $t_0 = 0$.

The smaller $\Delta = \max[\Delta_0, \cdots, \Delta_{N-1}]$, the higher the accuracy of the above formula. For small Δ, the summation in the exponent recalls, by its form, the Darboux sum corresponding to the integral

$$-\frac{1}{2}\int_0^T \left[\dot{x} - a(x,t)\right]^2 \frac{dt}{b(x,t)}, \qquad \left(x = x(t)\right) \tag{3}$$

(Here and in the sequel a dot denotes the time derivative.)

It is natural to inquire whether one can assign some precise significance to such an integral, and not just a symbolic one.

The realizations of $x(t)$ almost certainly do not have finite derivatives and, a fortiori, the latter are not square-integrable. Moreover the consideration of functionals of the type (3) is rather interesting from the point of view of applications, since in practice, as a rule, the realizations of a diffusion process are not exactl[illegible]-kov, but smooth ones with a finite derivative. For such processes an

---

* Translation of Proc. Sixth All-Union Conf. Theory Prob. and Math. Statist. (Vilnius, 1960), Gosudarstv. Izdat. Političesk. i Naučn. Lit. Litovsk. SSR, Vilnius, 1962, pp. 471–482. (Russian) MR 34 #5164.

AMS 1970 *subject classifications.* Primary 60J60; Secondary 60F05.

integral of the type (3) has an exact meaning.

One can conduct a systematic study of such smooth processes and their functionals by an explicit introduction to the theory of the operation of smoothing. However a simpler approach is also of interest, one without an explicit consideration of smoothing but which deals with functionals of smooth functions. For such functions one can take, within a known approximation, observed smoothed realizations.

A useful step in that direction is the introduction of the probability functional $W[x(t)]$ defined on the space B of functions $x(t)$ having a bounded continuous derivative $\dot{x}(t)$ of bounded variation.

1. Let $z(t)$ be a Wiener process with initial condition $z(0) = z_0$, described by the equation

$$\frac{\partial p(z, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(z, t)}{\partial z^2}. \tag{4}$$

The multivariate density (2) is defined in this case by the exact equality

$$p(z_1, \ldots, z_N \mid z_0) = \text{const} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{N-1} \left(\frac{z_{i+1} - z_i}{\Delta_i} \right)^2 \Delta_i \right\}. \tag{5}$$

It is natural to define the probability functional by the formula

$$W\left[z(t)\right] = \exp\left\{ -\frac{1}{2} \int_0^T \left[\dot{z}(t)\right]^2 dt \right\}. \tag{6}$$

Both in this case and in more general cases, the probability functional is defined only up to an arbitrary (finite) constant factor. We shall choose this factor in such a way as to obtain the simplest possible expression.

Let the functions $z(t) \in B$ for which the functional (6) is defined fulfill the conditions

$$\left|\dot{z}(t)\right| < M_z, \qquad 0 \leqslant t \leqslant T; \tag{7}$$

$$\sum_k \left| \dot{z}(\tau_{k+1}) - \dot{z}(\tau_k) \right| < M_z. \tag{8}$$

(Here $\cdots < \tau_k < \tau_{k+1} < \cdots$ are points at which $z(t)$ takes extremal values.) Condition (8) may be replaced by the inequality

(9)
$$\int_0^T |z|\, dt < M_z'.$$

Such replacement is always permissible without further restrictions if the derivative $\dot{z}$ and its integral are understood in the generalized sense.

THEOREM. 1. *Let* $z^{(1)}(t), z^{(2)}(t), h(t)$ *belong to B, and let* $z^{(1)}(0) = z^{(2)}(0) = z_0$, $h(t) > 0, 0 \leqslant t \leqslant T$ *and* $\epsilon > 0$. *Then*

(10)
$$\lim_{\varepsilon \to 0} \frac{P\{ z^{(1)} < z < z^{(1)} + \varepsilon h,\ 0 \leqslant t \leqslant T\}}{P\{ z^{(2)} < z < z^{(2)} + \varepsilon h,\ 0 \leqslant t \leqslant T\}} = \frac{W[z^{(1)}]}{W[z^{(2)}]}.$$

PROOF. Let us make the change of variables

(11)
$$\tilde{z}(t) = z(t) + z^{(2)}(t) - z^{(1)}(t).$$

The last process is described by the equation

$$\frac{\partial p(\tilde{z}, t)}{\partial t} = -(\dot{z}^{(2)} - \dot{z}^{(1)}) \frac{\partial p}{\partial \tilde{z}} + \frac{1}{2} \frac{\partial^2 p}{\partial \tilde{z}^2}.$$

According to the results of [1] the processes $z(t)$ and $\tilde{z}(t)$ are absolutely continuous, and the corresponding functional derivative is equal to

$$\frac{d\mu_{\tilde{z}}[z(t)]}{d\mu_z[z(t)]} = \exp\left\{ \int [\dot{z}^{(2)} - z^{(1)}]\, dz(t) - \frac{1}{2} \int [\dot{z}^{(2)} - z^{(1)}]^2\, dt \right\}$$

(12)
$$= \exp\left\{ -\frac{1}{2} \int_0^T \dot{z}^{(1)^2}\, dt + \frac{1}{2} \int_0^T \dot{z}^{(2)^2}\, dt + I \right\},$$

where

$$I = \int_0^T [z^{(2)} - \dot{z}^{(1)}]\, d[z - z^{(2)}].$$

By the Radon-Nikodým theorem we have

(13)
$$\mu_z(\Lambda_\varepsilon) = \int_{\Lambda_\varepsilon} \frac{d\mu_{\tilde{z}}}{d\mu_z}\, d\mu_z.$$

Here

$$\Lambda_\varepsilon = \left\{ z(t) : z^{(2)} < z(t) < z^{(2)} + \varepsilon h,\quad 0 \leqslant t \leqslant T;\quad z(0) = z_0 \right\}.$$

Substituting (12) into (13), we find

$$\frac{P\{z^{(1)} < z < z^{(1)}+\varepsilon h,\ 0\leqslant t\leqslant T\}}{P\{z^{(2)} < z < z^{(2)}+\varepsilon h,\ 0\leqslant t\leqslant T\}} = \frac{\mu_{\tilde z}(\Lambda_\varepsilon)}{\mu_z(\Lambda_\varepsilon)}$$

$$(14)\qquad = \exp\left\{-\frac{1}{2}\int_0^T \dot z^{(1)2}\,dt + \frac{1}{2}\int_0^T \dot z^{(2)2}\,dt\right\}\frac{1}{\mu_z(\Lambda_\varepsilon)}\int_{\Lambda_\varepsilon} e^I\,d\mu_z .$$

In the expression

$$I = \int_0^T [\dot z^{(2)} - \dot z^{(1)}]\,d[z - z^{(2)}] = [\dot z_T^{(2)} - \dot z_T^{(1)}]\,[z_T - z_T^{(2)}] - \int_0^T [z - z^{(2)}]\,[\ddot z^{(2)} - \ddot z^{(1)}]\,dt$$

one can, according to inequalities (7), (9), carry out the following estimates

$$\left|z - z^{(2)}\right| < \varepsilon h < \varepsilon M_h,\quad 0\leqslant t\leqslant T \quad \text{for}\quad z(t)\in\Lambda_\varepsilon;$$

$$\int \left|z - z^{(2)}\right|\left|\ddot z^{(2)} + \ddot z^{(1)}\right| dt < \varepsilon M_h \int \left|\ddot z^{(2)} - \ddot z^{(1)}\right| dt$$

$$\leqslant \varepsilon M_h \int \left[\left|\ddot z^{(2)}\right| + \left|\ddot z^{(1)}\right|\right] dt < \varepsilon M_h\left(M'_{z^{(2)}} + M'_{z^{(1)}}\right);$$

$$\left|\dot z_T^{(2)} - \dot z_T^{(1)}\right| < M_{z^{(1)}} + M_{z^{(2)}},$$

from which follows

$$\left|I\right| < \varepsilon M_h\left[M_{z^{(1)}} + M_{z^{(2)}} + M'_{z^{(1)}} + M'_{z^{(2)}}\right] \equiv \varepsilon M,$$

for all $z(t)\in\Lambda_\epsilon$. Therefore $|e^I - 1| < \epsilon M + \frac{1}{2}\epsilon^2 M^2 + \cdots$ and

$$\lim_{\varepsilon\to 0}\frac{1}{\mu_z(\Lambda_\varepsilon)}\int_{\Lambda_\varepsilon}(e^I - 1)\,d\mu_z = 0,$$

i. e.

$$(15)\qquad \lim_{\varepsilon\to 0}\frac{1}{\mu_z(\Lambda_\varepsilon)}\int_{\Lambda_\varepsilon} e^I\,d\mu_z = 1.$$

As a consequence of this and of (14), after passing to the limit as $\epsilon \to 0$, (10) follows.

2. Consider the Markov diffusion process $z(t)$, $0 < t < T$, corresponding to the equation

$$(16)\qquad \frac{\partial p(z,t)}{\partial t} = -\frac{\partial}{\partial z}\left[m(z,t)\,p\right] + \frac{1}{2}\frac{\partial^2 p}{\partial z^2},$$

where $m(z, t)$ is a bounded funct
with respect to z. The probability
satisfies the same equation (16), bu
$(z - z_1)$. Let us transform this eq
defined by the equality

(17) $p_{t_1 t}(z_1\ z) =$

where $f(z, t)$ is a function which
calculations lead to the equation

(18) $\frac{\partial \tilde p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial z^2} - \frac{\partial}{\partial z}\left[\left(m + \frac{\partial f}{\partial z}\right)\right.$

Choose the function $f(z, t)$ s
to zero, i. e. set

(19) $\frac{\partial f(z,t)}{\partial z} = m(z, t),$

The solution of the resulting e

(20) $\frac{\partial \tilde p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial z^2} +$

may be written, as we know, in the

(21) $p_{t_1 t_2}(z_1, z_2) = \int_{C_{z_1 z_2}} \exp\left\{-\right.$

with respect to the conditional Wien
Here and in the sequel

$C_{z_1 \ldots z_k} = \left\{ z(t) : \right.$

According to (17), the multiva

$p(z_1, \ldots, z_N \mid z_0) = p_{0 t_1}$

can be written in the form

$p(z_1, \ldots, z_T \mid z_0) = e^{f(z_0, 0) - f}$

if we take (21) into account,

where $m(z, t)$ is a bounded function with uniformly continuous first derivative with respect to z. The probability density $p_{t_1 t}(z_1, z)$ of jump from z_1 to z satisfies the same equation (16), but with initial condition $p_{t_1 t_1}(z_1, z) = \delta(z - z_1)$. Let us transform this equation, going over to the function $\tilde{p}_{t_1 t}(z_1, z)$ defined by the equality

$$p_{t_1 t}(z_1, z) = e^{f(z_1, t)}\, \tilde{p}_{t_1 t}(z_1, z)\, e^{-f(z, t)}, \tag{17}$$

where $f(z, t)$ is a function which will be defined explicitly in the sequel. Direct calculations lead to the equation

$$\frac{\partial \tilde{p}}{\partial t} = \frac{1}{2}\frac{\partial^2 \tilde{p}}{\partial z^2} - \frac{\partial}{\partial z}\left[\left(m + \frac{\partial f}{\partial z}\right)\tilde{p}\right] + \left[m\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial z^2} + \frac{1}{2}\left(\frac{\partial f}{\partial z}\right)^2\right]\tilde{p}. \tag{18}$$

Choose the function $f(z, t)$ such that the term with first derivative reduces to zero, i. e. set

$$\frac{\partial f(z, t)}{\partial z} = m(z, t), \quad f(z, t) = -\int_0^z m(z', t)\, dz'. \tag{19}$$

The solution of the resulting equation

$$\frac{\partial \tilde{p}}{\partial t} = \frac{1}{2}\frac{\partial^2 \tilde{p}}{\partial z^2} + \left[\frac{\partial f}{\partial t} - \frac{1}{2}m^2 - \frac{1}{2}\frac{\partial m}{\partial z}\right]\tilde{p} \tag{20}$$

may be written, as we know, in the form of the functional integral

$$\tilde{p}_{t_1 t_2}(z_1, z_2) = \int_{C_{z_1 z_2}} \exp\left\{-\frac{1}{2}\int_{t_1}^{t_2}\left[m^2 + \frac{\partial m}{\partial z} - 2\frac{\partial f}{\partial t}\right] dt\right\} dw[z \mid z_0] \tag{21}$$

with respect to the conditional Wiener measure $w[\Lambda \mid z_1] = \mathrm{P}(\Lambda \mid z(t_1) = z_1)$. Here and in the sequel

$$C_{z_1 \ldots z_k} = \left\{ z(t) : z(t_i) = z_i, \quad i = 1, \ldots, k \right\}.$$

According to (17), the multivariate distribution

$$p(z_1, \ldots, z_N \mid z_0) = p_{0t_1}(z_0, z_1) \ \ldots \ p_{t_{N-1} t_N}(z_{N-1}, z_N)$$

can be written in the form

$$p(z_1, \ldots, z_T \mid z_0) = e^{f(z_0, 0) - f(z_T, T)}\, \tilde{p}_{0t_1}(z_0, z_1) \ldots \tilde{p}_{t_{N-1}, T}(z_{N-1}, z_T),$$

or, if we take (21) into account,

$$p(z_1, \ldots, z_T \mid z_0) = e^{f(z_0,\,0) - f(z_T,\,T)} \times$$

$$\times \int\limits_{C_{z_1 \ldots z_T}} \exp\left\{ -\frac{1}{2} \int_0^T \left[m^2 + \frac{\partial m}{\partial z} - 2\, \frac{\partial f}{\partial t} \right] dt \right\} dw\,[z \mid z_0]. \tag{22}$$

Integrating the last expression, we find

$$\mu\,[\Lambda \mid z_0] = \int\limits_{\Lambda} \exp\left\{ f(z_0,\, 0) - f(z_T,\, T) \right.$$

$$\left. -\frac{1}{2} \int_0^T \left[m^2 + \frac{\partial m}{\partial z} - 2\, \frac{\partial f}{\partial t} \right] dt \right\} dw\,[z \mid z_0], \tag{23}$$

where

$$\Lambda = \Lambda_{t_1 \ldots t_N} \left\{ z(t) : z(t_i) \in E_i, \quad i = 1, \ldots, N \right\}.$$

Since in this formula $N, t_1, \cdots, t_N, E_1, \cdots, E_N$ are arbitrary, equation (23) holds for an arbitrary set Λ of more general form by virtue of the separability and continuity of the process under consideration. Therefore the measures $\mu\,[\Lambda \mid z_0]$ and $w\,[\Lambda \mid z_0]$ are absolutely continuous, and the corresponding functional derivative is equal to

$$\frac{d\mu\,[z \mid z_0]}{dw\,[z \mid z_0]} = \exp\left\{ f(z_0,\, 0) - f(z_T,\, T) + \int_0^T \frac{\partial f}{\partial t}\, dt - \frac{1}{2} \int_0^T \left[m^2 + \frac{\partial m}{\partial z} \right] dt \right\}. \tag{24}$$

Taking into account that $d\mu\,[z] = \mathrm{P}(dz_0)\, d\mu\,[z \mid z_0]$ and assuming the existence of an initial probability density $p_0(z_0) = \mathrm{P}(dz_0)/dz_0$, we have

$$d\mu\,[z] = p_0(z_0) \exp\left\{ f(z_0,\, 0) - f(z_T,\, T) + \int_0^T \frac{\partial f}{\partial t}\, dt \right.$$

$$\left. -\frac{1}{2} \int_0^T \left[m^2 + \frac{\partial m}{\partial z} \right] dt \right\} dw\,[z \mid z_0]\, dz_0. \tag{25}$$

Since the probability functional (6) corresponds to the Wiener measure $w\,[\Lambda \mid z_0]$, it is natural then, as is seen from (25), to define the probability functional of the given process $z(t)$ by the formula

$$W\left[z(t)\right]=p_0(z_0)\exp\left\{f(z_0,\ 0)-f(z_T,\ T)+\int_0^T\frac{\partial f}{\partial t}\,dt\right.$$

$$\left.-\frac{1}{2}\int_0^T\left[m^2+\frac{\partial m}{\partial z}\right]dt-\frac{1}{2}\int_0^T\dot{z}^2\,dt\right\}. \tag{26}$$

Theorem 1 in turn holds for this functional. Its proof can be carried out in a fashion similar to that of the previous proof, with some unessential complications. After substitution of the process (11), the equation for $z(t)$ takes the form

$$\frac{\partial p(\tilde{z},t)}{\partial t}=-\frac{\partial}{\partial\tilde{z}}\left\{\left[m(\tilde{z}-z^{(2)}+z^{(1)},\ t)+\dot{z}^{(2)}-\dot{z}^{(1)}\right]p\right\}+\frac{1}{2}\frac{\partial^2 p}{\partial z^2}. \tag{27}$$

Applying formula (24) to this case, we find

$$\frac{d\mu_{\tilde{z}}[\tilde{z}\mid\tilde{z}_0]}{dw[\tilde{z}\mid\tilde{z}_0]}=\exp\left\{f(z_0,\ 0)-f(z_T,\ T)+\int_0^T\left[\frac{\partial f(z,t)}{\partial t}+\frac{\partial f(z,t)}{\partial z}\left(-\dot{z}^{(2)}+\dot{z}^{(1)}\right)\right]dt\right.$$

$$+\int_0^T[\dot{z}^{(2)}-\dot{z}^{(1)}]\,d\tilde{z}-\frac{1}{2}\int_0^T\left[m^2(z,\ t)+\frac{\partial m(z,t)}{\partial z}\right]dt$$

$$\left.-\int_0^T m(z,\ t)\,[\dot{z}^{(2)}-\dot{z}^{(1)}]\,dt-\frac{1}{2}\int_0^T[\dot{z}^{(2)}-\dot{z}^{(1)}]^2\,dt\right\} \tag{28}$$

with

$$z=\tilde{z}-z^{(2)}+z^{(1)}.$$

Expressing $d\mu_{\tilde{z}}[\tilde{z}]/d\mu_z(\tilde{z})$ as the ratio of (28) to (24), we find

$$\frac{d\mu_{\tilde{z}}[\tilde{z}]}{d\mu_z[\tilde{z}]}=\frac{p_0\left(\tilde{z}_0-z_0^{(2)}+z_0^{(1)}\right)}{p_0(\tilde{z}_0)}\exp\left\{f(\tilde{z}_0-z_0^{(2)}+z_0^{(1)},\ 0)-f(\tilde{z}_T-z_T^{(1)}+z_T^{(2)},\ T)\right.$$

$$+\int_0^T\Phi(\tilde{z}-z^{(2)}+z^{(1)})\,dt-f(\tilde{z}_0,\ 0)+f(\tilde{z}_T,\ T)-\int_0^T\Phi(\tilde{z})\,dt$$

$$\left.-\frac{1}{2}\int_0^T\dot{z}^{(1)2}\,dt+\frac{1}{2}\int_0^T\dot{z}^{(2)2}\,dt+\int_0^T[\dot{z}^{(2)}-\dot{z}^{(1)}]\,d[\tilde{z}-z^{(2)}]\right\}, \tag{29}$$

where

$$\Phi(z)=-\frac{1}{2}m^2(z,\ t)-\frac{1}{2}\frac{\partial m(z,t)}{\partial z}+\frac{\partial f(z,t)}{\partial t}.$$

If in (29) one replaces the functions $\Phi(\widetilde{z}-z^{(2)}+z^{(1)})$, $\Phi(\widetilde{z})$, $f(\widetilde{z}_0-z_0^{(2)}+z_0^{(1)})$ $\ln p_0(\widetilde{z}_0)$, $\cdots$ by the functions $\Phi(z^{(1)})$, $\Phi(z^{(2)})$, $f(z_0^{(1)})$, $\ln p_0(z_0^{(2)})$, $\cdots$, expression (29) becomes $W[z^{(1)}]/W[z^{(2)}]$. Let us find an upper estimate for the difference between the first group of functions and the second one. Since these functions are uniformly continuous in z, this difference can be made arbitrarily small (for ϵ sufficiently small) simultaneously for all trajectories $\tilde{z}(t) \in \Delta_\epsilon$. As an example let us take the integral

$$\int_0^T \left| \Phi(\tilde{z} - z^{(2)} + z^{(1)}) - \Phi(z^{(1)}) \right| dt$$

and let us show that it can be made smaller than any $\delta > 0$. As a consequence of the uniform continuity we can find a μ such that

$$\left| \Phi(z - z^{(2)} + z^{(1)}) - \Phi(z^{(1)}) \right| < \frac{\delta}{T} \quad \text{for} \quad |\hat{z} - z^{(2)}| < \mu, \quad 0 \leqslant t \leqslant T.$$

Choosing $\epsilon = \mu/M_h$, we have

$$\int \left| \Phi(\tilde{z} - z^{(2)} + z^{(1)}) - \Phi(z^{(1)}) \right| dt < \delta; \quad \tilde{z}(t) \in \Lambda_\epsilon. \tag{30}$$

Therefore one can prove a limiting relation of the type (15) for the sum of all the terms in the exponent that give the difference between (29) and $W[z^{(1)}]/W[z^{(2)}]$, whence

$$\lim_{\epsilon \to 0} \frac{\mu_{\tilde{z}}[\Lambda_\epsilon]}{\mu_z[\Lambda_\epsilon]} = \frac{W[z^{(1)}]}{W[z^{(2)}]}.$$

For the special case when m is independent of t, when there exists a stationary distribution, and when it is taken as $p_0(z_0)$, the functional (26) is given by a formula that is symmetric in the sign of time:

$$W\left[z(t)\right] = \exp\left\{ -\frac{1}{2} \int_0^T \left[\dot{z}^2 + m^2(z) + \frac{\partial m(z)}{\partial z} \right] dt - f(z_0) - f(z_T) \right\},$$

which was found in [2].

Note that by virtue of the equality $df - (\partial f/\partial t)\, dt = (\partial f/\partial z)\, dz$ formula (26) may be written in the form

$$W\left[z(t)\right] = p_0(z_0) \exp\left\{ -\frac{1}{2} \int_0^T \left[(\dot{z} - m)^2 + \frac{\partial m}{\partial z} \right] dt \right\}. \tag{31}$$

3. Consider now the process $x(t)$ corresponding to the equation

$$\frac{\partial p_x}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x} \left[b(x, t) \frac{\partial p_x}{\partial x} \right], \tag{32}$$

where the function $b(x, t) = \sigma^2(x, t)$ is differentiable once with respect to t and twice with respect to x, and fulfills the conditions

$$0<\delta<b(x,\ t)<L.$$

It is a feature of equation (31) that its infinitesimal operator $\mathcal{L}=(dt/2)\,(\partial/\partial x)\,b(x,\ t)\,\partial/\partial x$ is symmetric, so that the Kolmogorov equations of the first and second kind coincide in form.

Let $x(t)$ satisfy the initial condition $x(0)=x_0$.

Define the probability functional $W[x(t)]$ on the space of functions $x(t)\in B$, satisfying the same condition $x(0)=x_0$, by means of the equality

$$\frac{W[x^{(1)}(t)]}{W[x^{(2)}(t)]}=\lim_{\varepsilon\to 0}\frac{P\{x^{(1)}<x<x^{(1)}+\varepsilon h\sigma(x^{(1)}),\ 0\leqslant t\leqslant T\}}{P\{x^{(2)}<x<x^{(2)}+\varepsilon h\sigma(x^{(2)}),\ 0\leqslant t\leqslant T\}},\tag{33}$$

when this limit exists and is independent of $h(t)>0$ of B. In order to calculate the ratio in the right-hand side of (33) let us make the change of variables

$$z(t)=\int_0^{x(t)}\frac{dx'}{\sigma(x',\ t)}\equiv Z\left(x(t),\ t\right).\tag{34}$$

By virtue of the conditions imposed on $b(x,\ t)$, this transform always exists and takes the function $x(t)\in B$ into a function $z(t)$ belonging to the same space B. Clearly

$$P\left\{x^{(i)}<x<x^{(i)}+\varepsilon h\sigma(x^{(i)}),\quad 0\leqslant t\leqslant T\right\}$$
$$=P\left\{z^{(i)}<z<z^{(i)}+\varepsilon h_i,\quad 0\leqslant t\leqslant T\right\},\qquad (i=1,\ 2),\tag{35}$$

where

$$z^{(i)}(t)=Z\left(x^{(i)}(t),\ t\right);\quad z^{(i)}(t)+h_i(t)=Z\left(x^{(i)}(t)+\varepsilon h(t)\,\sigma(x^{(i)}),\ t\right)$$

and therefore $h_i(t)=h(t)+O(\epsilon)$.

After the usual calculations related to the change of variable, we find that the probability density $p_z=p_x\,dx/dz$ corresponding to the new process $z(t)$ satisfies the equation

$$\frac{\partial p_z}{\partial t}=\frac{\partial}{\partial x}\left\{\left[\frac{1}{2}\ \frac{\partial\ln\sigma}{\partial z}+\frac{\partial Z(x,\ t)}{\partial t}\right]p_z\right\}+\frac{1}{2}\ \frac{\partial^2 p}{\partial z^2}.\tag{36}$$

If we now apply the results of the previous section, we obtain

$$\lim_{\varepsilon\to 0}\frac{P\{z^{(1)}<z<z^{(1)}+\varepsilon h,\ 0\leqslant t\leqslant T\}}{P\{z^{(2)}<z<z^{(2)}+\varepsilon h,\ 0\leqslant t\leqslant T\}}=\frac{W_z[z^{(1)}]}{W_z[z^{(2)}]},\tag{37}$$

$$W_z[z]=\exp\left\{-\frac{1}{2}\int_0^T\left[\left(z-\frac{1}{2\sigma}\frac{\partial\sigma}{\partial z}-\frac{\partial Z(x,t)}{\partial t}\right)^2+\frac{1}{2}\frac{\partial^2\ln\sigma}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{\partial Z(x,t)}{\partial t}\right)\right]dt\right\}.$$

By virtue of (33), (35), we have

$$W\left[x(t)\right]=\exp\left\{-\frac{1}{2}\int_0^T\left[\left(\dot z-\frac{1}{2}\frac{\partial\sigma}{\partial x}-\frac{\partial Z(x,t)}{\partial t}\right)^2+\frac{1}{2}\sigma\frac{\partial^2\sigma}{\partial x^2}+\sigma\frac{\partial^2 Z(x,t)}{\partial x\,\partial t}\right]dt\right\}$$

where we must set $z(t)=Z(x(t),t)$, i. e.

$$W\left[x(t)\right]=\exp-\frac{1}{2}\int_0^T\left[\left(\frac{\dot x}{\sigma}-\frac{1}{2}\frac{\partial\sigma}{\partial x}\right)^2+\sigma\frac{\partial}{\partial t}\left(\frac{1}{\sigma}\right)dt\right\}$$

$$=\left(\frac{\sigma(x_T,T)}{\sigma(x_0,0)}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\int_0^T\left[\frac{\dot x^2}{\sigma^2}+\frac{1}{4}\left(\frac{\partial\sigma}{\partial x}\right)^2+\frac{1}{2}\sigma\frac{\partial^2\sigma}{\partial x^2}\right]dt\right\}$$

$$=\left[\frac{b(x_T,T)}{b(x_0,0)}\right]^{\frac{1}{4}}\exp\left\{-\frac{1}{2}\int_0^T\left[\frac{\dot x^2}{b}-\frac{1}{16}\frac{1}{b}\left(\frac{\partial b}{\partial x}\right)^2+\frac{1}{4}\frac{\partial^2 b}{\partial x^2}\right]dt\right\} \tag{38}$$

4. Finally let us consider the general case of a one-dimensional diffusion process $x(t)$ having the infinitesimal operator

$$dL(t)=\frac{1}{2}b(x,t)\frac{\partial^2}{\partial x^2}+a(x,t)\frac{\partial}{\partial x}, \tag{39}$$

where $b(x,t)$, $a(x,t)$ are functions with properties similar to those mentioned above.

If we set its transition probability equal to

$$p_{t_1t}(x_1,x)=e^{f(x_1,t_1)}\tilde p_{t_1t}(x_1,x)\,e^{-f(x,t)},$$

the infinitesimal operator

$$d\tilde L=e^{-f(x,t)}\,dL\,e^{f(x,t)}+\frac{\partial f(x,t)}{\partial t}\,dt, \tag{40}$$

will correspond to the function $\tilde p_{t_1t}(x_1,x)$, or, substituting (39)

$$\frac{d\tilde L}{dt}=\frac{1}{2}\frac{\partial}{\partial x}b\frac{\partial}{\partial x}+\left[b\frac{\partial f}{\partial x}-\frac{1}{2}\frac{\partial b}{\partial x}+a\right]\frac{\partial}{\partial x}+a\frac{\partial f}{\partial x}+\frac{1}{2}b\left[\left(\frac{\partial f}{\partial x}\right)^2+\frac{\partial^2 f}{\partial x^2}\right]+ \tag{41}$$

PROBABILITY FUNCTIC

To make the term with the first

By analogy with §2, one can prove

respect to the auxiliary process

$(\partial/\partial x)\,b\,\partial/\partial x$. By virtue of (39

The probability functional $W[x($

bution density $(p_0(x_0)$ multiplied

indicated auxiliary process $x(t)$

Substituting (42), we transform thi

Introducing the notation $m=\sigma^{-1}($

functional in the following form:

This expression coincides with (31)

$p_0(x_0)$ by $\dot z$, $\partial m/\partial z$, $p_0(z_0)$. It

To make the term with the first dervative vanish, let us set

$$\frac{\partial f}{\partial t}=\frac{1}{2b}\,\frac{\partial b}{\partial x}-\frac{a}{b}. \tag{42}$$

By analogy with §2, one can prove that the process $x(t)$ is absolutely continuous with respect to the auxiliary process $\widetilde{x}(t)$ determined by the infinitesimal operator $\frac{1}{2}(\partial/\partial x)\, b\, \partial/\partial x$. By virtue of (39)–(42) the functional derivative is equal to

$$\frac{d\mu_z[x\mid x_0]}{d\mu_{\tilde z}[x\mid x_0]}=\exp\left\{ f(x_0,\ 0)-f(x_T,\ T)+\int_0^T \frac{\partial f}{\partial t}\,dt+\int_0^T a\,\frac{\partial f}{\partial x}\,dt+\frac{1}{2}\int_0^T b\left[\left(\frac{\partial f}{\partial x}\right)^2+\frac{\partial^2 f}{\partial x^2}\right]dt\right\}. \tag{43}$$

The probability functional $W[x(t)]$ is defined as the product of the initial distribution density ($p_0(x_0)$ multiplied by $\sigma(x_0, 0)$), the probability functional of the indicated auxiliary process $x(t)$ (38), and the functional (43):

$$W\left[x(t)\right]=\sqrt{\sigma(x_0,\ 0)\ \sigma(x_T,\ T)}\ p_0(x_0)\ \exp\left\{ f(x_0,\ 0)-f(x_T,\ T)+\int_0^T \frac{\partial f}{\partial t}\,dt-\frac{1}{2}\int_0^T\left[\frac{\dot x^2}{\sigma^2}+\frac{1}{4}\left(\frac{\partial\sigma}{\partial x}\right)^2+\frac{1}{2}\,\sigma\,\frac{\partial^2\sigma}{\partial x^2}\right]dt+\int_0^T a\,\frac{\partial f}{\partial x}\,dt+\frac{1}{2}\int_0^T b\left[\left(\frac{\partial f}{\partial x}\right)^2+\frac{\partial^2 f}{\partial x^2}\right]dt\right\}.$$

Substituting (42), we transform this expression into the form

$$W\left[x(t)\right]=\sqrt{\sigma(x_0,\ 0)\ \sigma(x_T,\ T)}\ p_0(x_0)\ \exp\left\{-\frac{1}{2}\int_0^T\frac{\dot x^2}{b}\,dt+\int_0^T\left[a-\frac{1}{2}\,\frac{\partial b}{\partial x}\right]\frac{\dot x}{b}\,dt-\frac{1}{2}\int_0^T\left[\frac{a^2}{b}+b\,\frac{\partial}{\partial x}\left(\frac{a}{b}\right)-\frac{1}{4}\,b\,\frac{\partial}{\partial x}\left(\frac{1}{b}\,\frac{\partial b}{\partial x}\right)-\frac{1}{16}\,\frac{1}{b}\left(\frac{\partial b}{\partial x}\right)^2\right]dt\right\}. \tag{44}$$

Introducing the notation $m=\sigma^{-1}(a-\frac{1}{4}(\partial b/\partial x))$, we can write the probability functional in the following form:

$$W\left[x(t)\right]=\sigma(x_0,\ 0)\ p_0(x_0)\ \exp\left\{-\frac{1}{2}\int_0^T\left[\frac{\dot x^2}{b}-2\,\frac{m\dot x}{\sigma}+m^2+\sigma\,\frac{\partial m}{\partial x}\right]dt\right\}. \tag{45}$$

This expression coincides with (31) if we replace $\dot x/\sigma$, $\sigma\partial m/\partial x$, $\sigma(x_0, 0)\, p_0(x_0)$ by $\dot z$, $\partial m/\partial z$, $p_0(z_0)$. It follows from this that Theorem 1 (which is proved with the help of the substitution of variables (34)) holds for functional (44).

It is clear from the above that Theorem 1 may be taken as the definition of the probability functional. Then from this definition follow the rules for transforming the functional by a change of variables and the formulas for the relation of the probability functionals of processes that are mutually absolutely continuous. Thus we have

THEOREM 2. *If the processes* $x(t), y(t),\ 0 \leqslant t \leqslant T$, *are absolutely continuous and if the corresponding functional derivative has the form*

$$\frac{d\mu_x[x]}{d\mu_y[x]} = \exp\left\{ F\left(x(0),\ x(T)\right) + \int_0^T \Phi\left(x(t),\ t\right) dt \right\},$$

where $F(x, x')$, $\Phi(x, t)$ *are bounded functions the first of which is continuous and the second is uniformly continuous in* x, *and if they have probability functionals* $W_x[x]$, $W_y[y]$, *then*

$$\frac{W_x[x]}{W_y[x]} = \exp\left\{ F\left(x(0),\ x(T)\right) + \int_0^T \Phi\left(x(t),\ t\right) dt \right\}, \qquad x(t) \in R. \tag{46}$$

For the proof, one must take into consideration that by definition of the probability functionals

$$\frac{W_x[x^{(1)}]}{W_y[x^{(1)}]}\,\frac{W_y[x^{(2)}]}{W_x[x^{(2)}]} = \lim_{\varepsilon\to 0}\frac{\mu_x\left(\Lambda_\varepsilon^{(1)}\right)}{\mu_x\left(\Lambda_\varepsilon^{(2)}\right)}\cdot\lim_{\varepsilon\to 0}\frac{\mu_y\left(\Lambda_\varepsilon^{(2)}\right)}{\mu_y\left(\Lambda_\varepsilon^{(1)}\right)},\ \left(\Lambda_\varepsilon^{(i)} = \left\{x(t) : x^{(i)} < x < x^{(i)} + \ldots\right.\right.$$

and therefore

$$\frac{W_x[x^{(1)}]}{W_y[x^{(1)}]}\,\frac{W_y[x^{(2)}]}{W_x[x^{(2)}]} = \lim_{\varepsilon\to 0}\frac{\mu_x\left(\Lambda_\varepsilon^{(1)}\right)}{\mu_y\left(\Lambda_\varepsilon^{(1)}\right)}\quad\frac{\mu_y\left(\Lambda_\varepsilon^{(2)}\right)}{\mu_x\left(\Lambda_\varepsilon^{(2)}\right)}. \tag{47}$$

Making use of the continuity hypothesis, we can easily verify that the differences

$$F\left(x(0),\ x(T)\right) - F\left(x^{(i)}(0),\ x^{(i)}(T)\right);\quad \int_0^T \Phi(x,\ t)\ dt - \int_0^T \Phi(x^{(i)},\ t)\ dt$$

can be made, for ϵ sufficiently small, arbitrary small simultaneously for all $x(t) \in \Lambda_\epsilon^{(i)}$ by analogy with (30). Therefore an equality of the type (15) holds for these differences, and we have the convergence

$$\lim_{\varepsilon\to 0}\frac{\mu_x\left(\Lambda_\varepsilon^{(i)}\right)}{\mu_y\left(\Lambda_\varepsilon^{(i)}\right)} = \exp\left\{ F\left(x^{(i)}(0),\ x^{(i)}(T)\right) + \int_0^T \Phi(x^{(i)},\ t)\ dt \right\}. \tag{48}$$

(47) we replace the limit of the r
the ratio of the limits lim $(\mu_x(\Lambda_\epsilon^{(1)})$
and formula (46), in accordance wi

It is important to note that unde
behaves like a scalar, and not as

Comparing (44) with (3), we see
gral (3), there appeared a considerat

All of the above can be generalize
process. Let $x(t) = \{x_1(t), \cdots, x$
nitesimal operator

$$\frac{dL}{dt} = \frac{1}{2}\, b_{\alpha\beta}(x,\ t)$$

equation

$$\frac{\partial p(x,\ t)}{\partial t} = \frac{1}{2}\ \frac{\partial}{\partial x_\alpha}$$

The functions $a(x, t), b_{\alpha\beta}(x, t) =$
derivatives $\partial a_\alpha/\partial x_\gamma$, $\partial^2 b_{\alpha\beta}/\partial x_\gamma \partial x$

$$0 < \delta < \sigma^2(x,\ t) < L;\quad \sigma(x,$$

The probability functional $W[x(t)]$
by the formula

$$\frac{W[x^{(1)}(t)]}{W[x^{(2)}(t)]} = \lim_{\varepsilon\to 0}\frac{P\{x}{P\{x}$$

Here $S_\epsilon^{(i)}(t)$ is a sphere in R_n con
gue measure

$$L\left[S_\varepsilon^{(i)}(t)\right] = \varepsilon h(t)\ \sigma\left(x^{(i)}(t),\ t\right);\quad 0$$

If we take $b_{\alpha\beta}(x, t)$ as the metric
the volume of domain $S_\epsilon^{(i)}(t)$ equal
Calculations lead to the following ex

$$W\left[x(t)\right] = \sigma\left(x(0),\ 0\right) p_0\left(x(0)\right.$$

$$-2m_\alpha(\sigma^{-1})_{\alpha\beta}\,\dot{x}_\beta + m_\alpha m$$

In (47) we replace the limit of the ratio $\mu_x(\Lambda_\epsilon^{(1)})/\mu_y(\Lambda_\epsilon^{(1)}) : \mu_x(\Lambda_\epsilon^{(2)})/\mu_y(\Lambda_\epsilon^{(2)})$ by the ratio of the limits $\lim (\mu_x(\Lambda_\epsilon^{(1)})/\mu_y(\Lambda_\epsilon^{(1)}))$ and $\lim (\mu_x(\Lambda_\epsilon^{(2)})/\mu_y(\Lambda_\epsilon^{(2)}))$, we find formula (46), in accordance with (48).

It is important to note that under a change of variables the probability functional behaves like a scalar, and not as a scalar density.

Comparing (44) with (3), we see that after we gave a precise meaning to the integral (3), there appeared a considerable number of additional terms.

All of the above can be generalized to the case of a multidimensional diffusion process. Let $x(t) = \{x_1(t), \cdots, x_n(t)\}$ be such a process with corresponding infinitesimal operator

$$\frac{dL}{dt} = \frac{1}{2}\, b_{\alpha\beta}(x,\ t)\, \frac{\partial^2}{\partial x_\alpha\, \partial x_\beta} + a_\alpha(x,\ t)\, \frac{\partial}{\partial x_\alpha}$$

and equation

(49)
$$\frac{\partial p(x,\ t)}{\partial t} = \frac{1}{2}\, \frac{\partial^2}{\partial x_\alpha\, \partial x_\beta}\, [b_{\alpha\beta}\, p] - \frac{\partial}{\partial x_\alpha}\, [a_\alpha\, p].$$

The functions $a(x, t)$, $b_{\alpha\beta}(x, t) = \sigma_{\alpha\gamma}\sigma_{\gamma\beta}$ are such that there exist continuous derivatives $\partial a_\alpha/\partial x_\gamma$, $\partial^2 b_{\alpha\beta}/\partial x_\gamma \partial x_\delta$, $\partial b_{\alpha\beta}/\partial t$, and

(50)
$$0 < \delta < \sigma^2(x,\ t) < L; \quad \sigma(x,\ t) = \mathrm{Det}\, \left\| \sigma_{\alpha\beta} \right\| = \mathrm{Det}^{\frac{1}{2}} \left\| b_{\alpha\beta} \right\|.$$

The probability functional $W[x(t)]$, where $x(t) \in B$; $i = 1, \cdots, n$, is defined by the formula

(51)
$$\frac{W[x^{(1)}(t)]}{W[x^{(2)}(t)]} = \lim_{\varepsilon \to 0} \frac{P\left\{ x(t) \in S_\varepsilon^{(1)}(t),\ 0 \leqslant t \leqslant T \right\}}{P\left\{ x(t) \in S_\varepsilon^{(1)}(t),\ 0 \leqslant t \leqslant T \right\}}.$$

Here $S_\epsilon^{(i)}(t)$ is a sphere in R_n containing $x^{(i)}(t)$, $i = 1, 2$, and having Lebesgue measure

(52)
$$L\left[S_\varepsilon^{(i)}(t)\right] = \varepsilon h(t)\, \sigma\left(x^{(i)}(t),\ t\right); \quad 0 < h(t) \in B; \quad i = 1,\ 2; \quad 0 \leqslant t \leqslant T.$$

If we take $b_{\alpha\beta}(x, t)$ as the metric tensor in the space R_n, then we must take the volume of domain $S_\epsilon^{(i)}(t)$ equal to $\epsilon h(t)$.

Calculations lead to the following expression for the probability functional

(53)
$$W\left[x(t)\right] = \sigma\left(x(0),\ 0\right) p_0\left(x(0)\right) \exp\left\{ -\frac{1}{2} \int_0^T \left[\dot{x}_\alpha (b^{-1})_{\alpha\beta}\, x_\beta - 2m_\alpha (\sigma^{-1})_{\alpha\beta}\, \dot{x}_\beta + m_\alpha m_\alpha + \sigma_{\alpha\beta} \frac{\partial m_\alpha}{\partial x_\beta} \right] dt \right\},$$

where

$$m_\alpha = (\sigma^{-1})_{\alpha\beta}\left(a_\beta - \frac{1}{2}\,\frac{\partial \sigma_{\beta\gamma}}{\partial x_\delta}\,\sigma_{\gamma\delta}\right), \tag{54}$$

and $p_0(x)$ is the initial probability density; in formulas (49), (53), (54), summation is carried out over repeated indices. *

Translated by:
A. N. Rossolimo

* Editor's note. The bibliography has been omitted in the original.