Logic and Constraint Satisfaction

An Introduction

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A Primer on Logic

- What is logic?

A Primer on Logic

- What is logic?
- “Logic is logic. That’s all I say."

  The Deacon’s Masterpiece

  Oliver Wendell Holmes, Sr., 1858

Outline

1. Basic Notions
2. Conjunctive Queries & CSP
3. Datalog and CSP
4. Finite-variable Logics and CSP

Vocabularies and Structures

Definition

- *Vocabulary* σ: a set $\sigma = \{R'_1, \ldots, R'_m\}$ of relation symbols of specified arities.
- *σ-structure* $\mathbf{A} = (A, R_1, \ldots, R_m)$: a non-empty set A and relations on A such that $\mathrm{arity}(R_i) = \mathrm{arity}(R'_i)$, $1 \leq i \leq m$.
- *Finite σ-structure* $\mathbf{A}$: universe A is finite

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Example

- *Graph:* $\mathbf{G} = (V, E)$, where E is binary.
- *String:* $\mathbf{S} = (\{1, 2, \ldots, n\}, P)$, where P is unary
  $m \in P \iff$ the m-th bit of the string is 1.
  - String 10001 encoded as $(\{1, 2, 3, 4, 5\}, \{1, 5\})$.

Vocabularies and Structures

Example

Every 3-CNF formula can be viewed as a finite structure of the form $\mathbf{A} = (A, R_0, R_1, R_2, R_3)$, where each R_i is a 3-ary relation.

- 3-CNF formula φ with variables $x_1, \ldots, x_n$
- Structure $\mathbf{A}^\varphi = (\{x_1, \ldots, x_n\}, R_0^\varphi, R_1^\varphi, R_2^\varphi, R_3^\varphi)$, where

$$
\begin{aligned}
R_0^\varphi &= \{(x, y, z) : (x \vee y \vee z) \text{ is a clause of } \varphi\} \\
R_1^\varphi &= \{(x, y, z) : (\neg x \vee y \vee z) \text{ is a clause of } \varphi\} \\
R_2^\varphi &= \{(x, y, z) : (\neg x \vee \neg y \vee z) \text{ is a clause of } \varphi\} \\
R_3^\varphi &= \{(x, y, z) : (\neg x \vee \neg y \vee \neg z) \text{ is a clause of } \varphi\}
\end{aligned}
$$

Queries

Definition

- *Class $\mathcal{C}$ of structures:* a collection of relational σ-structures closed under isomorphisms.
- *k-ary Query Q on $\mathcal{C}$*, where $k \geq 1$:
  a mapping Q with domain $\mathcal{C}$ and such that
  - $Q(\mathbf{A})$ is a k-ary relation on $\mathbf{A}$, for $\mathbf{A} \in \mathcal{C}$;
  - Q is *preserved under isomorphisms*, i.e.,
    if $h : \mathbf{A} \to \mathbf{B}$ is an isomorphism, then $Q(\mathbf{B}) = h(Q(\mathbf{A}))$.
- *Boolean Query Q on $\mathcal{C}$:*
  a mapping $Q : \mathcal{C} \to \{0, 1\}$ preserved under isomorphisms.
  Thus, Q can be identified with the subclass $\mathcal{C}'$ of $\mathcal{C}$, where
  $$\mathcal{C}' = \{\mathbf{A} \in \mathcal{C} : Q(\mathbf{A}) = 1\}.$$

Examples of Queries

- PATH OF LENGTH 2: Binary query on graphs $\mathbf{H} = (V, E)$

  $P2(\mathbf{H}) = \{(a, b) \in V^2\text{: there is a path of length 2 from } a \text{ to } b\}.$

- CONNECTIVITY: Boolean query on graphs $\mathbf{H} = (V, E)$

$$CN(\mathbf{H}) = \begin{cases} 1 & \text{if } \mathbf{H} \text{ is connected} \\ 0 & \text{otherwise.} \end{cases}$$

- k-COLORABILITY, $k \geq 2$
- 3-SAT (with formulas viewed as structures)

Definability of Queries

Definition

Let L be a logic and $\mathcal{C}$ a class of structures

- A k-ary query Q on $\mathcal{C}$ is L-*definable* if there is an L-formula $\varphi(x_1, \ldots, x_k)$ with $x_1, \ldots, x_k$ as free variables and such that for every $\mathbf{A} \in \mathcal{C}$

$$Q(\mathbf{A}) = \{(a_1, \ldots, a_k) \in A^k : \mathbf{A} \models \varphi(a_1, \ldots, a_k)\}.$$

- A Boolean query Q on $\mathcal{C}$ is L-*definable* if there is an L-sentence ψ such that for every $\mathbf{A} \in \mathcal{C}$

$$Q(\mathbf{A}) = 1 \iff \mathbf{A} \models \psi.$$

First-Order Logic

Definition

First-Order Logic FO (on graphs):

- *first-order variables:* $x, y, z, \ldots$
- *atomic formulas:* $E(x, y)$, $x = y$
- *formulas:* atomic formulas, Boolean connectives, first-order quantifiers $\exists x$, $\forall x$, $\exists y$, $\forall y$, $\ldots$ that range over the nodes of the graph.

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Example

On the class $\mathcal{G}$ of finite graphs the query PATH OF LENGTH 2 is FO-definable

$$P2(\mathbf{H}) \; = \; \{(a, b) \in V^2 : \mathbf{H} \models \exists z(E(a, z) \wedge E(z, b))\}.$$

Limitations of First-Order Logic

Fact

- The queries TRANSITIVE CLOSURE, CONNECTIVITY, k-COLORABILITY, $k \geq 2$, are not FO-definable.
- On the class of all finite structures with 4 ternary relations, the query 3-SAT is not first-order definable.

Note: Results about non-definability in FO-logic can be proved using Ehrenfeucht-Fraïssé games.

The Complexity of Logic

Definition (Vardi, 1982)

- The *combined complexity* of L is the following decision problem:
  Given a finite structure **B** and an L-sentence ψ, does **B** $\models \psi$?

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- The *data complexity* of L is the family of the following decision problems P_ψ, one for each fixed L-sentence ψ:
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- The *data complexity* of L is the family of the following decision problems P_ψ, one for each fixed L-sentence ψ:
  Given a finite structure $\mathbf{B}$, does $\mathbf{B} \models \psi$?
- The *expression complexity* of L is the family of the following decision problems $P_{\mathbf{B}}$, one for each fixed finite structure $\mathbf{B}$:
  Given an L-sentence ψ, does $\mathbf{B} \models \psi$?

Some Basic Complexity Classes

Definition

- L: problems solvable by a TM in logspace
- NL: problems solvable by a NTM in logspace
- P: problems solvable by a TM in polynomial time
- NP: problems solvable by a NTM in polynomial time
- PSPACE: problems solvable by a TM in polynomial space.

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- PSPACE: problems solvable by a TM in polynomial space.

Fact

- $\mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} = \mathrm{NPSPACE}$.
- $\mathrm{NL} \neq \mathrm{PSPACE}$
- No other proper containment is known at present.
- Each of them possesses natural complete problems.

The Complexity of Logic

Definition

Let L be a logic and C a complexity class.

- *The data complexity of L is in C* if for each L-sentence ψ, the problem P_ψ is in C.
- *The data complexity of L is C-complete* if it is in C and there is at least one L-sentence ψ such that P_ψ is C-complete.

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- *The data complexity of L is C-complete* if it is in C and there is at least one L-sentence ψ such that P_ψ is C-complete.
- *The expression complexity of L is in C* if for each finite structure **B**, the problem $P_{\mathbf{B}}$ is in C.
- *The expression complexity of L is C-complete* if it is in C and there is at least one finite structure **B** such that $P_{\mathbf{B}}$ is C-complete.

The Complexity of First-Order Logic

Theorem

- The data complexity of FO is in L.
- Both the expression complexity of FO and the combined complexity of FO are PSPACE-complete.

Proof.

- Fix a first-order sentence ψ. Given a finite structure **B**: Cycle through all possible instantiations of the quantifiers of ψ in **B**, keeping track of the number of them using a counter in binary.
- QBF is PSPACE-complete (Stockmeyer - 1976).

$\square$

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1 Basic Notions

2 Conjunctive Queries & CSP

3 Datalog and CSP

4 Finite-variable Logics and CSP

Conjunctive Queries

Definition

- A *primitive positive formula (pp-formula)* is a FO-formula in prenex normal form built from atomic formulas, $\wedge$, and $\exists$ only, i.e., it is of the form:
$$(\exists z_1 \ldots \exists z_m)\psi(x_1, \ldots, x_k, z_1, \ldots, z_m),$$
where ψ is a conjunction of atomic formulas.
- A *conjunctive query* is a query definable by a pp-formula.

Conjunctive Queries

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- A *conjunctive query* is a query definable by a pp-formula.

Example

- PATH OF LENGTH 2 (Binary query)
$$(\exists z)(E(x_1, z) \wedge E(z, x_2))$$
Can also be written as a rule:
$$P2(x_1, x_2) \ :-\ E(x_1, z), E(z, x_2)$$

Conjunctive Queries and Databases

Fact

- Conjunctive queries are the most frequently asked queries in databases (a.k.a. SPJ queries)
- The main construct of SQL expresses conjunctive queries.

Example

Relations $R_1(A, B, C)$, $R_2(B, C, D)$

```
SELECT  R1.A, R2.D
  FROM  R1, R2
 WHERE  R1.B = R2.B AND R1.C = R2.C
```

This expresses the conjunctive query $Q(x, w)$ definable by

$$\exists y \exists z (R_1(x, y, z) \wedge R_2(y, z, w))$$

Conjunctive Query Evaluation

A fundamental problem about conjunctive queries

Definition

CONJUNCTIVE QUERY EVALUATION

- Given a CQ Q and a structure $\mathbf{A}$, find
$$Q(\mathbf{A}) = \{(a_1, \ldots a_k) : \mathbf{A} \models Q(a_1, \ldots, a_k)\}$$
- For Boolean queries Q, this becomes:
Given Q and $\mathbf{A}$, does $\mathbf{A} \models Q$? (is $Q(\mathbf{A}) = 1$?)
- Same problem as the *combined complexity of pp-formulas*.

Example

- Given a graph $\mathbf{H}$, does it contain a triangle?

Conjunctive Query Containment

Another fundamental problem about conjunctive queries

Definition

CONJUNCTIVE QUERY CONTAINMENT

- Given two k-ary CQs Q_1 and Q_2, is it true that for every structure $\mathbf{A}$, we have that $Q_1(\mathbf{A}) \subseteq Q_2(\mathbf{A})$?
- For Boolean queries, this becomes:
  Given two Boolean queries Q_1 and Q_2, does $Q_1 \models Q_2$?
  (does Q_1 logically imply Q_2?)

Example

Is it true that if two nodes of a graph $\mathbf{H}$ are connected by a path of length 4, then they are also connected by a path of length 3?

Conjunctive Queries and Homomorphisms

- Chandra and Merlin (1977) showed that:

  CONJUNCTIVE QUERY EVALUATION
  and
  CONJUNCTIVE QUERY CONTAINMENT
  are the *same* problem.

Conjunctive Queries and Homomorphisms

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  are the *same* problem.

- The link is the HOMOMORPHISM PROBLEM:

  Given **A** and **B**, does $\mathbf{A} \longrightarrow \mathbf{B}$?

Canonical CQs and Canonical Structures

Definition

- Given $\mathbf{A} = (A, R_1^{\mathbf{A}}, \ldots, R_m^{\mathbf{A}})$, the *canonical CQ of* $\mathbf{A}$ is the Boolean CQ $Q^{\mathbf{A}}$ with the elements of A as variables and the “facts" of $\mathbf{A}$ as conjuncts: $\exists x_1 \cdots \exists x_n (\bigwedge_{i=1}^{m} \bigwedge_{\mathbf{t}} R_i^{\mathbf{A}}(\mathbf{t}))$
- If Q is a Boolean CQ, then $\mathbf{A}^Q$ is the structure with the variables of Q as elements and the conjuncts of Q as “facts".

Example

- If $\mathbf{A} = (\{a,b,c\}, \{(a,b),(b,c),(c,a)\}$, then $Q^{\mathbf{A}}$ is
$$\exists x \exists y \exists z (E(x,y) \wedge E(y,z) \wedge E(z,x))$$
- If Q is $\exists x \exists y (E(x,y) \wedge E(x,z))$, then
$$\mathbf{A}^Q = (\{x,y,z\}, \{(x,y),(x,z)\})$$

Homomorphisms, CQE and CQC

Theorem (Chandra and Merlin, 1977)

For all relational structures $\mathbf{A}$ and $\mathbf{B}$, the following statements are equivalent:

1. $\mathbf{A} \longrightarrow \mathbf{B}$
2. $\mathbf{B} \models Q^{\mathbf{A}}$
3. $Q^{\mathbf{B}} \subseteq Q^{\mathbf{A}}$.

Homomorphisms, CQE and CQC

Alternatively,

Theorem (Chandra and Merlin, 1977)

For all conjunctive queries Q_1 and Q_2, the following statements are equivalent:

1. $Q_1 \subseteq Q_2$
2. $\mathbf{A}^{Q_2} \longrightarrow \mathbf{A}^{Q_1}$
3. $\mathbf{A}^{Q_1} \models Q_2$.

Illustration: 3-COLORABILITY

Example

For a graph $\mathbf{H}$, the following are equivalent:

1. $\mathbf{H} \rightarrow \mathbf{K_3}$ (i.e., $\mathbf{H}$ is 3-colorable)
2. $\mathbf{K}_3 \models Q^{\mathbf{H}}$
3. $Q^{\mathbf{K}_3} \subseteq Q^{\mathbf{H}}$

(1) $\Longrightarrow$ (2): A hom. $h : \mathbf{H} \rightarrow \mathbf{K_3}$ provides witnesses in $\mathbf{K}_3$ for the $\exists$ quantifiers in $Q^{\mathbf{H}}$.

(2) $\Longrightarrow$ (3): If $\mathbf{K}_3 \models Q^{\mathbf{H}}$ and $\mathbf{A} \models Q^{\mathbf{K}_3}$, then there are witness functions $h : \mathbf{H} \rightarrow \mathbf{K_3}$ and $h^* : \mathbf{K_3} \rightarrow \mathbf{A}$. Then $h^* \circ h : \mathbf{H} \rightarrow \mathbf{A}$ provides witnesses in $\mathbf{A}$ for the $\exists$ quantifiers in $Q^{\mathbf{H}}$.

(3) $\Longrightarrow$ (1): Since $\mathbf{K}_3 \models Q^{\mathbf{K}_3}$, we have $\mathbf{K}_3 \models Q^{\mathbf{H}}$. The witnesses to the $\exists$ quantifiers give a homomorphism from $\mathbf{H}$ to $\mathbf{K}_3$.

CSP, Homomorphisms, CQE, and CQC

Fact

- CONSTRAINT SATISFACTION
- THE HOMOMORPHISM PROBLEM
- CONJUNCTIVE QUERY EVALUATION
- CONJUNCTIVE QUERY CONTAINMENT

are the same problem.

CSP, Homomorphisms, CQE, and CQC

Fact

- The combined complexity of conjunctive queries (pp-formulas) coincides with the HOMOMORPHISM PROBLEM (UNIFORM CSP).

- The expression complexity of conjunctive queries (pp-formulas) coincides with the family of problems $\mathrm{CSP}(\mathbf{B})$, where

$$\mathrm{CSP}(\mathbf{B}) \;=\; \{\mathbf{A} : \mathbf{A} \to \mathbf{B}\} \;=\; \{\mathbf{A} : \mathbf{B} \models Q^{\mathbf{A}}\}.$$

- Both the combined complexity and the expression complexity of conjunctive queries are NP-complete. (contrast with FO.)

Tractability of CSP via Logic

Fact

- The complexity of $\mathrm{CSP}(\mathbf{B})$ depends on $\mathbf{B}$:
  - $\mathrm{CSP}(\mathbf{K}_3)$ is 3-COLORABILITY, hence is NP-complete.
  - $\mathrm{CSP}(\mathbf{K}_2)$ is 2-COLORABILITY, hence is in P.

Approach

- Use logic to identify tractable (polynomial-time solvable) cases of $\mathrm{CSP}(\mathbf{B})$.
- Study when $\mathrm{CSP}(\mathbf{B})$ is definable in some logic L whose data complexity is in P.

CSP and Unions of Conjunctive Queries

Definition

For every structure $\mathbf{B}$, let

$$\neg\mathrm{CSP}(\mathbf{B}) \;=\; \{\mathbf{A} : \mathbf{A} \not\to \mathbf{B}\}.$$

Fact

For every structure $\mathbf{B}$:

- $\neg\mathrm{CSP}(\mathbf{B})$ is closed under homomorphisms.
- Moreover,

$$\neg\mathrm{CSP}(\mathbf{B}) \;=\; \{\mathbf{A} : \mathbf{A} \models \bigvee_{\mathbf{D} \not\to \mathbf{B}} Q^{\mathbf{D}}\},$$

  i.e., $\neg\mathrm{CSP}(\mathbf{B})$ is definable by an infinite union of conjunctive queries.

CSP and Unions of Conjunctive Queries

Definition

- $L_{\infty\omega}$ is the extension of FO with infinitary disjunctions and infinitary conjunctions.
- $\exists L^{+}_{\infty\omega}$ is the existential positive fragment of $L_{\infty\omega}$.

Approach

- Thus, for every structure $\mathbf{B}$, we have that $\neg\mathrm{CSP}(\mathbf{B})$ is $\exists L^{+}_{\infty\omega}$-definable, since

$$\neg\mathrm{CSP}(\mathbf{B}) \; = \; \{\mathbf{A} : \mathbf{A} \models \bigvee_{\mathbf{D} \not\to \mathbf{B}} Q^{\mathbf{D}}\}.$$

- Study when $\neg\mathrm{CSP}(\mathbf{B})$ is definable in a tractable fragment of $\exists L^{+}_{\infty\omega}$.

CSP and First-Order Logic

Fact

Assume that **B** is a structure such that $\neg\mathrm{CSP}(\mathbf{B})$ is definable by a finite union of conjunctive queries (i.e., $\neg\mathrm{CSP}(\mathbf{B}) = \bigvee_{i=1}^{m} Q^{\mathbf{D}_i}$). Then $\mathrm{CSP}(\mathbf{B})$ is FO-definable; hence, it is in P.

CSP and First-Order Logic

Fact

Assume that **B** is a structure such that $\neg\mathrm{CSP}(\mathbf{B})$ is definable by a finite union of conjunctive queries (i.e., $\neg\mathrm{CSP}(\mathbf{B}) = \bigvee_{i=1}^{m} Q^{\mathbf{D}_i}$). Then $\mathrm{CSP}(\mathbf{B})$ is FO-definable; hence, it is in P.

Theorem (Atserias, 2005)

For every structure **B**, the following statements are equivalent.

1. $\mathrm{CSP}(\mathbf{B})$ is FO-definable.
2. $\neg\mathrm{CSP}(\mathbf{B})$ is definable by a finite union of conjunctive queries.

CSP and First-Order Logic

Example (Gallai-Hesse-Roy Theorem, circa 1965)

Let $\mathbf{T}_k$ be the linear order with k elements and $\mathbf{P}_{k+1}$ be the directed path with $k+1$ elements. Then, for every directed graph $\mathbf{G}$, we have that

$$\mathbf{G} \rightarrow \mathbf{T}_k \iff \mathbf{P}_{k+1} \not\rightarrow \mathbf{G}.$$

Consequently,

$$\neg\mathrm{CSP}(\mathbf{T}_k) = \{\mathbf{G} : \mathbf{G} \models Q^{\mathbf{P}_{k+1}}\}.$$

Beyond First-Order Logic

Fact

- $\mathrm{CSP}(\mathbf{K}_2)$ is 2-Colorability.
- $\mathrm{CSP}(\mathbf{K}_2)$ is in P, but it is not FO-definable.
- Hence, $\neg\mathrm{CSP}(\mathbf{K}_2)$ is definable by an infinite union of conjunctive queries, but it is not definable by any finite union of conjunctive queries.

Question

Can the tractability of $\mathrm{CSP}(\mathbf{K}_2)$ be explained via definability in a logic other than FO?

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Tractability via Definability in Datalog

Fact (Feder and Vardi, 1993)

Definability of $\neg\mathrm{CSP}(\mathbf{B})$ in Datalog is a unifying explanation for many tractability results about $\mathrm{CSP}(\mathbf{B})$, including $\mathrm{CSP}(\mathbf{K}_2)$.

Datalog

Note: Recall that every CQ can be written as a rule:

$$P2(x_1, x_2) \ :- \ E(x_1, z), E(z, x_2)$$

Definition

- Datalog $=$ Conjunctive Queries $+$ Recursion
  Function, negation-free, and $\neq$-free logic programs
- A Datalog program is a finite set of rules given by conjunctive queries

$$T(\overline{x}) \ :- \ S_1(\overline{y}_1), \ldots, S_r(\overline{y}_r).$$

  Intensional DB predicates (IDBs): those that occur both in the *heads* and the *bodies* of rules (recursive predicates).

  Extensional DB predicates (EDBs): all other predicates.

Example (Transitive Closure Query TC)

$$TC(\mathbf{H}) = \{(a,b) : \text{there is a path from } a \text{ to } b \text{ in } \mathbf{H}\}.$$

A Datalog program for TC (linear Datalog)

$$\begin{array}{|lcl} S(x,y) & :- & E(x,y) \\ S(x,y) & :- & E(x,z), S(z,y) \end{array}$$

Another Datalog program for TC (non-linear Datalog)

$$\begin{array}{|lcl} S(x,y) & :- & E(x,y) \\ S(x,y) & :- & S(x,z), S(z,y) \end{array}$$

- E is the EDB.
- S is the IDB; it defines TC.

Semantics of Datalog Programs

Example

A Datalog program for TC

$$\left| \begin{array}{lcl} S(x,y) & :- & E(x,y) \\ S(x,y) & :- & E(x,z), S(z,y) \end{array} \right.$$

Operational Semantics: "Bottom-up" Evaluation

$$\left| \begin{array}{lcl} S^0 & = & \emptyset \\ S^{m+1} & = & \{(a,b)) : \exists z(E(a,z) \wedge S^m(z,b))\} \end{array} \right.$$

Fact: The following statements are true:

$$\begin{array}{lcl} S^m & = & \{(a,b) : \text{there is a path of length} \leq m \text{ from } a \text{ to } b\} \\ TC & = & \bigcup_m S^m \; = \; S^{|V|}. \end{array}$$

Datalog and 2-Colorability

Example

- $\mathrm{CSP}(\mathbf{K}_2) = $ 2-COLORABILITY.
- Recall that a graph is 2-colorable if and only if it does not contain an odd cycle.
- Datalog program for NON-2-COLORABILITY:

$$\left|\begin{array}{lll} O(X,Y) & :- & E(X,Y) \\ O(X,Y) & :- & O(X,Z), E(Z,W), E(W,Y) \\ Q & :- & O(X,X) \end{array}\right.$$

Data Complexity of Datalog

Theorem

- Every Datalog query is definable by an “effective and uniform" union of conjunctive queries.
- Every Datalog query is in P.
- The data complexity of Datalog is P-complete.

Proof.

- The “bottom-up" evaluation of Datalog programs converges in polynomially-many steps.
- Each iteration is definable by a finite union of conjunctive queries.
- HORN 3-UNSAT is P-complete and expressible in Datalog.

$\square$

Horn 3-SAT and Datalog

Fact (Horn 3-UnSat is expressible in Datalog)

- Horn 3-CNF formula φ viewed as a finite structure
  $$\mathbf{A}^{\varphi} = (\{x_1, \ldots, x_n\}), U, P, N), \quad \text{where}$$
  - U is the set of unit clauses x
  - P is the set of clauses $(\neg x \vee \neg y \vee z)$
  - N is the set of clauses $(\neg x \vee \neg y \vee \neg z)$.
- Datalog program for Horn 3-UnSat: encodes the unit propagation algorithm for Horn Satisfiability.

$$\left|\begin{array}{lll} T(z) & :- & U(z) \\ T(z) & :- & P(x,y,z), T(x), T(y) \\ Q & :- & N(x,y,z), T(x), T(y), T(z) \end{array}\right.$$

Provably non-linearizable.

Tractability via Definability in Datalog

Fact (Feder and Vardi, 1993)

Definability of $\neg\mathrm{CSP}(\mathbf{B})$ in Datalog is a unifying explanation for many tractability results about $\mathrm{CSP}(\mathbf{B})$.

Theorem (Feder and Vardi, 1993)

- If $\mathbf{B} = (B, R_1, \ldots, R_k)$ is such that $\mathrm{Pol}(\{R_1, \ldots, R_k\})$ contains a near-unanimity function, then $\neg\mathrm{CSP}(\mathbf{B})$ is Datalog-definable.

  Special Case: 2-SAT

- If $\mathbf{B} = (B, R_1, \ldots, R_k)$ is such that $\mathrm{Pol}(\{R_1, \ldots, R_k\})$ contains a semi-lattice function, then $\neg\mathrm{CSP}(\mathbf{B})$ is Datalog-definable.

  Special Case: HORN k-SAT, DUAL HORN k-SAT, $k \geq 2$.

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Logics with Finitely Many Variables

An old, but fruitful, idea: the number of variables is a resource.

Definition

- FO^k: FO-formulas with at most k variables.
- L^k: FO^k-formulas built from atomic formulas, $\wedge$, and $\exists$ only.
- Note: Each L^k-formula defines a conjunctive query.

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- Note: Each L^k-formula defines a conjunctive query.

Example

- $P^n(x, y)$: there is a path of length n from x to y.

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- FO^k: FO-formulas with at most k variables.
- L^k: FO^k-formulas built from atomic formulas, $\wedge$, and $\exists$ only.
- Note: Each L^k-formula defines a conjunctive query.

Example

- $P^n(x, y)$: there is a path of length n from x to y.
- $P^n(x, y)$ is L^3-definable.

$$
\begin{aligned}
P^1(x,y) &\equiv E(x,y) \\
P^{n+1}(x,y) &\equiv \exists z(E(x,z) \wedge \exists x((x = z) \wedge P_n(x,y))).
\end{aligned}
$$

k-Datalog

Definition

A k-Datalog program is a Datalog program in which each rule $t_0 \; :- \; t_1, \ldots, t_m$ has at most k distinct variables.

Example

- NON 2-COLORABILITY revisited

$$
\left|
\begin{array}{lcl}
O(X,Y) & :- & E(X,Y) \\
O(X,Y) & :- & O(X,Z), E(Z,W), E(W,Y) \\
Q & :- & O(X,X)
\end{array}
\right.
$$

- Therefore, NON 2-COLORABILITY is definable in 4-Datalog.
- Exercise: NON 2-COLORABILITY is definable in 3-Datalog.

Datalog and Finite-Variable Logics

Theorem (K ... and Vardi, 1990)

- Every k-Datalog definable query is also definable by a formula of the form $\bigvee_{n\geq 1} \psi_n$, where ψ_n is an L^k-formula.
- Consequently, k-Datalog $\subseteq \exists L^{k,+}_{\infty\omega}$.

Note

In general, k-Datalog is a proper fragment of $\exists L^{k,+}_{\infty\omega}$.
(The latter can express non-recursive queries using arbitrary infinite disjunctions.)

Datalog, Finite-Variable Logics, and CSP

Theorem (K ... and Vardi, 1998)

For every **B** and every $k \geq 1$, the following are equivalent:

1. $\neg\mathrm{CSP}(\mathbf{B})$ is definable in k-Datalog.
2. $\neg\mathrm{CSP}(\mathbf{B})$ is definable by a formula of the form $\bigvee_{n\geq 1} \psi_n$, where each ψ_n is an L^k-formula.
3. $\neg\mathrm{CSP}(\mathbf{B})$ is definable in $\exists L^{k,+}_{\infty\omega}$.

Note

Recall that

$$\neg\mathrm{CSP}(\mathbf{B}) = \{\mathbf{A} : \mathbf{A} \models \bigvee_{\mathbf{D}\not\to\mathbf{B}} Q^{\mathbf{D}}\}$$

and each $Q^{\mathbf{D}}$ is a conjunctive query.

CSP and Logic

Summary

For every structure **B** and for every $k \geq 1$:

- $\neg\mathrm{CSP}(\mathbf{B})$ is definable by an (infinite) union of conjunctive queries.
- $\neg\mathrm{CSP}(\mathbf{B})$ is FO-definable if and only if it is definable by a finite union of conjunctive queries.
- $\neg\mathrm{CSP}(\mathbf{B})$ is definable in k-Datalog if and only if it is definable by an (infinite) union of conjunctive queries each of which is L^k-definable.

Existential k-Pebble Games

Spoiler and Duplicator play on two structures **A** and **B**. Each player uses k pebbles, labeled $1, \ldots, k$. In each move,

- Spoiler places a pebble on or removes a pebble from an element of **A**.
- Duplicator tries to duplicate the move on **B** using the pebble with the same label.

$$\begin{array}{llllll} \mathbf{A}: & a_1 & a_2 & \ldots & a_l & \\ & \downarrow & \downarrow & \cdots & \downarrow & \\ \mathbf{B}: & b_1 & b_2 & \ldots & b_l & \quad l \leq k \end{array}$$

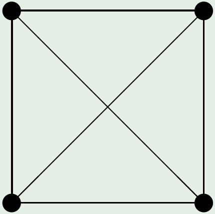
- Spoiler *wins* the $(\exists, k)$-pebble game if at some point the mapping $a_i \mapsto b_i$, $1 \leq i \leq l$, is not a partial homomorphism.
- Duplicator *wins* the $(\exists, k)$-pebble game if the above never happens.

Fact (Cliques of Different Size)

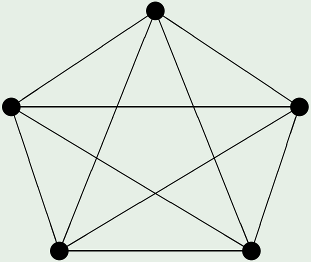
Let $\mathbf{K}_k$ be the k-clique. Then

- Duplicator wins the $(\exists, k)$-pebble game on $\mathbf{K}_k$ and $\mathbf{K}_{k+1}$.
- Spoiler wins the $(\exists, k)$-pebble game on $\mathbf{K}_k$ and $\mathbf{K}_{k-1}$.

Example



$\mathbf{K}_4$



$\mathbf{K}_5$

Winning Strategies in the $(\exists, k)$-Pebble Game

Definition

A *winning strategy* for the Duplicator in the $(\exists, k)$-pebble game is a non-empty set $\mathcal{I}$ of partial homomorphisms from **A** to **B** such that

- If $f \in \mathcal{I}$ and $h \subseteq f$, then $h \in \mathcal{I}$ ($\mathcal{I}$ is *closed under subfunctions*).
- If $f \in \mathcal{I}$ and $|f| < k$, then for every $a \in A$, there is $g \in \mathcal{I}$ so that $f \subseteq g$ and $a \in \mathrm{dom}(g)$. ($\mathcal{I}$ has the *forth property up to k*)

Fact

If $\mathbf{A} \rightarrow \mathbf{B}$, then, for every $k \geq 1$, the Duplicator wins the $(\exists, k)$-pebble game on **A** and **B**.

k-Datalog and $(\exists, k)$-Pebble Games

Theorem (K ... and Vardi)

- Let Q be a query definable in $\exists L^{k,+}_{\infty\omega}$. If **A** satisfies Q and the Duplicator wins the $(\exists, k)$-pebble game on **A** and **B**, then also **B** satisfies Q.
- There is a polynomial-time algorithm to decide whether, given two finite structures **A** and **B**, the Spoiler or the Duplicator wins the $(\exists, k)$-pebble game on **A** and **B**.
- For every fixed finite structure **B**, there is a k-Datalog program that expresses the query: given a finite structure **A**, does the Spoiler win the $(\exists, k)$-game on **A** and **B**?

k-Datalog, $\exists L^{k,+}_{\infty\omega}$, $(\exists, k)$-pebble games, and CSP

Theorem

Let k be a positive integer and **B** a finite structure. Then the following statements are equivalent:

1. $\neg\mathrm{CSP}(\mathbf{B})$ is definable in k-Datalog.
2. $\neg\mathrm{CSP}(\mathbf{B})$ is definable in $\exists L^{k,+}_{\infty\omega}$.
3. $\mathrm{CSP}(\mathbf{B}) = \{\mathbf{A} : \text{Duplicator wins the } (\exists, k)\text{-pebble game on } \mathbf{A} \text{ and } \mathbf{B}\}$.

Note

Single *canonical* polynomial-time algorithm for all $\mathrm{CSP}(\mathbf{B})$'s that are definable in k-Datalog:
Determine the winner in the $(\exists, k)$-pebble game.

The Hierarchy Problem for Datalog-definable CSPs

Problem

Prove or disprove:
For every $k \geq 4$, there is a directed graph $\mathbf{G}_k$ such that

- $\neg\mathrm{CSP}(\mathbf{G}_k)$ is definable in k-Datalog;
- $\neg\mathrm{CSP}(\mathbf{G}_k)$ is not definable in $(k-1)$-Datalog.

Note

- NON 2-COLORABILITY is definable in 3-Datalog, but not in 2-Datalog.
- All $\neg\mathrm{CSP}(\mathbf{G})$'s presently known to be definable in Datalog are actually definable in 3-Datalog.

The Meta-problem for Datalog-definable CSPs

Problem

Determine whether or not the following problems are decidable:

- Given a structure **B**, is $\neg\mathrm{CSP}(\mathbf{B})$ definable in Datalog?
- Given a structure **B**, is $\neg\mathrm{CSP}(\mathbf{B})$ definable in k-Datalog? (Here k is a fixed positive integer.)

Theorem (Larose, Loten, and Tardif, 2006)

The following problem is NP-complete:
Given a structure **B**, is $\mathrm{CSP}(\mathbf{B})$ definable in first-order logic?

Epilogue

- “Logic is in the eye of the logician."

  Outrageous Acts and Everyday Rebellions

  Gloria Steinem, 1986.