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9 Slender vortex motion

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Abstract

A fast numerical method based on the Klein–Knio's thin filament model is presented for computing the motion of slender vortex filaments. Combining the Biot-Savart law and an asymptotic expansion, Klein and Knio derived an equation for the evolution of a thin vortex filament. However, in their numerical discretization the spatial step is restricted by the overlapping condition of the traditional thin tube method. That is, the spatial step has to be small in comparison with the core of the vortex filament. The stiffness of the system is inversely proportional to the square of the spatial step. As a result, a small spatial step requires a tiny time step, thereby makes the simulations virtually impossible for thin filaments. Here we propose a fast numerical method to overcome this difficulty. In our approach, the spatial discretization does not depend on the core size. Our method significantly speeds up the computation for thin vortex filament.

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1. Introduction

Thin vortex filament model has an important application in the prediction of the behavior of the trailing vortices behind aircraft (Crow [1], Ting [2], Widnall [3]). Chorin [4] and Chorin and Akao [5] suggest that slender vortices play an important role in governing the structure of turbulent flows. In addition, superfluid vortex filaments are observed to be smooth and have very tiny core radius (~ 1Å) [6]. So the thin filament model fits perfectly with the superfluid vortex filaments. These are just a few of the ample motivations for the studies of thin vortex filament dynamics.

The irrotational flow surrounding a three-dimensional concentrated slender vortex is described by the well-known Biot-Savart law

$$\mathbf{v}(\mathbf{x}) = -\frac{\Gamma}{4\pi}\int_C \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \times d\mathbf{x}', \tag{1}$$

where C is the vortex centerline. This induced velocity field becomes singular as one approaches points on C. There are many approximations based on (1) to model slender vortex flow, including the self-induction approximation [7], the Klein-Majda equation [8], and vortex-element-based thin-tube method [9]. All these approaches simplify vortex core dynamics by either ignoring core structure or assuming a constant core. Klein and Knio [10] presented a thin filament model, which was originated from Callegari and Ting [11]. In this model, the core vorticity distribution is assumed to evolve slowly compared to the core rotation rate. In [10] they also developed a numerical scheme for the simulation of slender vortex filaments. The advantage of their approach is that the core structure evolution is included. However, their numerical scheme based on the vortex-element method is restricted by the overlapping condition, which could be very expensive and inefficient for thin vortex motion. In this paper, we develop a fast numerical method for Klein and Knio's thin filament model. Our method no longer requires the overlapping condition, thereby significantly speeds up the numerical simulations. Our numerical method can be easily extended to solve the Klein-Knio-Ting model [12] for viscous slender vortices.

The paper is organized as follows. We start with an introduction to the thin filament model, followed by a brief review of Klein and Knio's numerical method. Then we give a detailed description of our fast method. Numerical experiments are carried out for superfluid vortex motion.

2. Thin filament model

A thin vortex filament is a vorticity distribution that is highly concentrated in the vicinity of a smooth time-dependent curve $L(t) : s \to \mathbf{x}(s, t)$ as sketched in Fig. 1. The radius δ of the vortex core is much smaller than a characteristic radius of curvature R of $L(t) \cdot L(t)$ is usually called the centerline of the filament. However, this is not a very good definition for the centerline. According to this definition, any smooth curve $\tilde{L}(t)$ within $O(\delta^2)$ range of $L(t)$ is also a centerline. More important, $L(t)$ may not be a material trajectory of $\mathbf{x}(s, 0)$, i.e., $L(t)$ may not move with the fluid. Even if $L(t)$ moves with the fluid, the vorticity distribution observed relative to $L(t)$ and the motion of $L(t)$ may be changing very rapidly. As we will see below, if we follow an arbitrary centerline