AMS 212B, Final Exam, Spring 2009

1. Use the matched asymptotic expansion to solve the BVP (boundary value problem)

$$\begin{cases} \varepsilon y'' - y' + y^2 = 0 \\ y(0) = \dfrac{1}{2}, \quad y(1) = 0 \end{cases}, \qquad \varepsilon \to 0+$$

   Find <u>the leading term</u> in the composite expansion.

2. Use the method of strained variable to solve the IVP (initial value problem)

$$\begin{cases} y'' + y\left(1 + \varepsilon y^2\right) = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}, \qquad \varepsilon \to 0$$

   Find <u>the first two terms</u> of the asymptotic expansion.

   <u>Hint:</u>

   The solution of $\begin{cases} w'' + w = \cos(3s) \\ w(0) = 0, \quad w'(0) = 0 \end{cases}$ is $w(s) = \frac{1}{8}\left[\cos(s) - \cos(3s)\right]$.

   You may need to use $\cos^3(s) = \frac{3}{4}\cos(s) + \frac{1}{4}\cos(3s)$

3. Use Watson's lemma to obtain an asymptotic expansion of

$$\int_0^3 \exp\left(-\lambda t^2\right)\left(1 - \sin(t)\right) dt, \qquad \lambda \to +\infty$$

   Find <u>the first two terms</u> of the asymptotic expansion

   <u>Hint:</u>

   Watson's Lemma: $\int_0^\infty \exp(-\lambda s) s^\alpha ds = \dfrac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}$

   You may need to use $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4. Find <u>the leading term expansion</u> of

$$\int_{-1}^1 e^t \cos\left[\lambda\left(\cos(t)\right)\right] dt, \qquad \lambda \to +\infty$$

<u>Hint:</u> you may need the expansions of $\int_{-a}^{b} \cos\left(\lambda s^2\right) ds$ and $\int_{-a}^{b} \sin\left(\lambda s^2\right) ds$. If you do not remember these expansions, you can derive them from

$$\int_0^\infty \cos\left(x^2\right) dx = \int_0^\infty \sin\left(x^2\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$