AMS 212, Assignment #1

1. Use integration by parts to derive the asymptotic series of

   $$f(z) = 2e^{z^2}\int_z^{\infty} e^{-t^2}\,dt\,, \qquad z \to +\infty$$

   Find the first three terms.

   Hint:

   $$\int_z^{\infty} e^{-t^2}\,dt = \int_z^{\infty} \frac{1}{2t}\,d\left(-e^{-t^2}\right) = \cdots$$

2. Using rescaling to find asymptotic series of the three roots of

   $$\varepsilon x^3 + x^2 - 1 = 0\,, \qquad \varepsilon \to 0$$

   Find the first three terms of each root.

3. Find the asymptotic series of the root of

   $$e^{-x} - x - 1 + \varepsilon = 0\,, \qquad \varepsilon \to 0$$

   When $\varepsilon = 0$, $x = 0$ is the unperturbed root.

   Find the first two terms of the root.

   Hint: Use the expansion form

   $$x = \varepsilon a_1 + \varepsilon^2 a_2 + \cdots$$

   Or use the iterative method.

4. Design an iteration formula to solve

   $$\sin(x) - x + \frac{\varepsilon}{6} = 0\,, \qquad \varepsilon \to 0$$

   When $\varepsilon = 0$, $x = 0$ is the unperturbed root.

   Find the first two terms of the root.

   Hint: $\sin(x) - x = -\frac{x^3}{6} + \cdots$ does not have a linear term.

   The leading term above suggests us to write the equation as

$$\frac{x^3}{6} = \sin(x) - x + \frac{x^3}{6} + \frac{\varepsilon}{6}$$

==> $$x = \left(6\sin(x) - 6x + x^3 + \varepsilon\right)^{\frac{1}{3}}$$

5. (Optional) We study the accuracy of the asymptotic series

$$e^z \int_z^\infty \frac{e^{-x}}{x} dx \sim \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{z^n}$$

We examine the error

$$E(z,N) = \left| e^z \int_z^\infty \frac{e^{-x}}{x} dx - \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{z^n} \right|$$

a) For z = 10, plot $E(z, N)$ vs N (using the log scale for the error).

b) For each fixed value of z, find numerically

$$N(z) = \arg\min_N E(z,N)$$

Plot $N(z)$ vs z.

c) Then plot $E(z,N(z))$ vs z *using the log scale for the error).

Note that $E(z,N(z))$ is the minimum error (the best accuracy) we can achieve at z.

Hint: In Matlab, the exponential integral function is defined as

$$\text{expint}(z) = \int_z^\infty \frac{e^{-x}}{x} dx$$

So, in Matlab, we can calculate $e^z \int_z^\infty \frac{e^{-x}}{x} dx$ using

$$e^z \int_z^\infty \frac{e^{-x}}{x} dx = e^z \cdot \text{expint}(z)$$