A Distributed Receding Horizon Control Algorithm for Dynamically Coupled Nonlinear Systems

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***Abstract*— This paper presents a distributed receding horizon control law for dynamically coupled nonlinear systems that are subject to decoupled input constraints. The subsystem dynamics are modelled by continuous time nonlinear ordinary differential equations, and the coupling comes in the form of state dependence between subsystems. Examples of such systems include tight formations of aircraft and certain large scale process control systems. Given separable quadratic cost structure, a distributed controller is defined for each subsystem, and feasibility and asymptotic stabilization are established. Coupled subsystems communicate the previously computed trajectory at each receding horizon update. Key requirements for stability are that each distributed optimal control not deviate too far from the remainder of the previous optimal control, and that the amount of dynamic coupling is sufficiently small.**

Keywords: receding horizon control, model predictive control, distributed control, decentralized control, large scale systems.

I. INTRODUCTION

The problem of interest is to design a distributed controller for a set of dynamically coupled nonlinear subsystems that are required to perform a cooperative task. Examples of such situations where distributed control is desirable include tight formations of aircraft [2] and certain large scale process control systems [13]. The control approach advocated here is receding horizon control (RHC). In RHC, the current control action is determined by solving a finite horizon optimal control problem online at every update. In continuous time formulations, each optimization yields an open-loop control trajectory and the initial portion of the trajectory is applied to the system until the next update. A survey of RHC, also known as model predictive control, is given by Mayne *et al*. [9]. Advantages of RHC are that a large class of performance objectives, dynamic models and constraints can be transparently accommodated.

In this paper, subsystems that are dynamically coupled are referred to as *neighbors*. The work presented here is a continuation of a recent work [6], wherein a *distributed implementation* of RHC is presented in which neighbors are coupled solely through cost functions. As in [6], each subsystem here: is assigned its own optimal control problem, optimizes only for its own control at each update, and exchanges information only with neighboring subsystems. The motivation for pursuing such a distributed implementation is to enable the autonomy of the individual subsystems, respect the (decentralized) information constraints, and reduce the computational burden of centralized implementations.

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Previous work on distributed RHC of dynamically coupled systems include Jia and Krogh [7], Motee and Sayyar-Rodsaru [11] and Acar [1]. All of these papers address coupled LTI subsystem dynamics with quadratic separable cost functions. State and input constraints are not included, aside from a stability constraint in [7] that permits state information exchanged between the subsystems to be delayed by one update period. In these papers, analysis is facilitated by exploiting the analytic solutions available in the LTI case. In another work, Jia and Krogh [8] solve local min-max problems for coupled nonlinear subsystems, where the neighboring subsystem states are treated as bounded disturbances. Stability is obtained by contracting each subsystems state at every sample period, until the objective set is reached. As such, stability does not depend on information updates between neighbors, although such updates may improve performance. When subsystems are cooperative, it is anticipated that performance should improve by making more informed assumptions about neighboring subsystems, as done in the implementation here. In the future, this conjecture will be tested in numerical experiments.

We begin in Section II by defining the nonlinear coupled subsystem dynamics and control objective. In Section III, distributed optimal control problems are defined for each subsystem, and the distributed receding horizon control algorithm is defined. Feasibility and stability results are presented in Section IV, and Section V discusses conclusions and future work.

II. SYSTEM DESCRIPTION AND OBJECTIVE

In this section, the system dynamics and control objective are defined. For any vector $x \in \mathbb{R}^n$, $\|x\|_P$ denotes the P-weighted 2-norm, defined by $\|x\|_P^2 = x^T P x$, and P is any positive-definite real symmetric matrix. Also, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the largest and smallest eigenvalues of P, respectively. Often, the notation $\|x\|$ is understood to mean $\|x(t)\|$ at some instant of time $t \in \mathbb{R}$.

Our objective is to stabilize a group of $N_a \geq 2$ dynamically coupled agents toward the origin in a cooperative and distributed way using RHC. For each agent $i \in \{1, ..., N_a\}$, the state and control vectors are denoted $z_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$, respectively, at any time $t \geq t_0 \in \mathbb{R}$. The dimension of every agents state (control) are assumed to be the same, for notational simplicity and without loss of generality. The concatenated vectors are denoted $z = (z_1, ..., z_{N_a})$ and $u = (u_1, ..., u_{N_a})$.

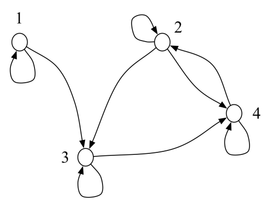
The dynamic coupling between the agents is topologically identified by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} =$

$\{1,...,N_a\}$ is the set of nodes (agents) and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of all directed edges between nodes in the graph. The set $\mathcal{E}$ is defined in the following way. If any components of z_j appear in the dynamic equation for agent i, for some $j \in \mathcal{V}$, it is said that j is an *upstream neighbor* of agent i, and $\mathcal{N}_i^{\mathrm{u}} \subseteq \mathcal{V}$ denotes the set of upstream neighbors of any agent $i \in \mathcal{V}$. The set of all directed edges is defined as $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid j \in \mathcal{N}_i^{\mathrm{u}}, \forall i \in \mathcal{V}\}$. For every $i \in \mathcal{V}$, it is assumed that z_i appears in the dynamic equation for i, and so $i \in \mathcal{N}_i^{\mathrm{u}}$ for every $i \in \mathcal{V}$. In the language of graph theory, then, every node has a self-loop edge in $\mathcal{E}$. Note that $j \in \mathcal{N}_i^{\mathrm{u}}$ does not necessarily imply $i \in \mathcal{N}_j^{\mathrm{u}}$.

It will also be useful to reference the set of agents for which any of the components of z_i arises in their dynamical equation. This set is referred to as the *downstream neighbors* of agent i, and is denoted $\mathcal{N}_i^{\mathrm{d}}$. The set of all directed edges can be equivalently defined as $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid i \in \mathcal{N}_j^{\mathrm{d}}, \forall i \in \mathcal{V}\}$. Note that $j \in \mathcal{N}_i^{\mathrm{u}}$ if and only if $i \in \mathcal{N}_j^{\mathrm{d}}$, for any $i,j \in \mathcal{V}$. Consider the example system and corresponding directed graph given in Figure 1.

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_3) \\ \dot{z}_2 &= f_2(z_2, z_3, z_4) \quad \Longrightarrow \\ \dot{z}_3 &= f_3(z_3, z_4) \\ \dot{z}_4 &= f_4(z_4, z_2) \end{aligned}$$

(a)



(b)

Fig. 1. Example of (a) a set of coupled dynamic equations and (b) the corresponding directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ associated with the directed information flow. In this example, $\mathcal{V} = \{1,2,3,4\}$ and $\mathcal{E} = \{(1,1),(1,3),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4),(4,2)\}$. The upstream neighbor sets are $\mathcal{N}_1^{\mathrm{u}} = \{1,3\}$, $\mathcal{N}_2^{\mathrm{u}} = \{2,3,4\}$, $\mathcal{N}_3^{\mathrm{u}} = \{3,4\}$ and $\mathcal{N}_4^{\mathrm{u}} = \{2,4\}$, and the downstream neighbor sets are $\mathcal{N}_1^{\mathrm{d}} = \{1\}$, $\mathcal{N}_2^{\mathrm{d}} = \{2,4\}$, $\mathcal{N}_3^{\mathrm{d}} = \{1,2,3\}$ and $\mathcal{N}_4^{\mathrm{d}} = \{2,3,4\}$. By this convention, arrows in the graph point upstream.

It is assumed in this paper that the graph $\mathcal{G}$ is connected. Consequently, for every $i \in \mathcal{V}$, the set $\left(\mathcal{N}_i^{\mathrm{d}} \cup \mathcal{N}_i^{\mathrm{u}}\right) \setminus \{i\} \neq \emptyset$, and every agent is dynamically coupled to at least one other agent. It is also assumed that agents can receive information directly from each and every upstream neighbor, and agents can transmit information directly to each and every downstream neighbor. The *coupled* time-invariant nonlinear system dynamics for each agent $i \in \mathcal{V}$ are given by

$$\dot{z}_i(t) = f_i(z_i(t), z_{-i}(t), u_i(t)), \quad t \geq t_0, \tag{1}$$

where $z_{-i} = (z_{j_1}, ..., z_{j_l})$, $l = |\mathcal{N}_i^{\mathrm{u}}|$, denotes the concatenated vector of the states of the upstream neighbors of i. Each agent i is also subject to the decoupled input constraints $u_i(t) \in \mathcal{U},\ t \geq t_0$. The set $\mathcal{U}^N$ is the N-times Cartesian product $\mathcal{U} \times \cdots \times \mathcal{U}$. In concatenated vector form, the system dynamics are

$$\dot{z}(t) = f(z(t), u(t)), \quad t \geq t_0, \tag{2}$$

given $z(t_0)$, and $f = (f_1, ..., f_{N_a})$.

Assumption 1: The following holds: (**a**) f is C^2 and $0 = f(0,0)$; (**b**) system (2) has a unique solution for any $z(t_0)$ and any piecewise right-continuous control $u : [t_0, \infty) \to \mathcal{U}^{N_a}$; (**c**) $\mathcal{U} \subset \mathbb{R}^m$ is compact, containing the origin in its interior; (**d**) for every $i \in \mathcal{V}$, there exist positive constants α_i, γ_i and β_i^j, $j \in \mathcal{N}_i^{\mathrm{u}}$, such that

$$\|f_i(z_i, z_{-i}, u_i)\|_{P_i} \leq \alpha_i \|z_i\|_{P_i} + \sum_{j \in \mathcal{N}_i^{\mathrm{u}}} \beta_i^j \|z_j\|_{P_i} + \gamma_i \|u_i\|_{W_i},$$

holds for all $z_i(t) \in \mathbb{R}^n$, $z_j(t) \in \mathbb{R}^n$, $j \in \mathcal{N}_i^{\mathrm{u}}$, and $u_i(t) \in \mathcal{U}$, $t \geq t_0$, and P_i, W_i will be defined in the next section.

Consider now the linearization of (1) around the origin, denoted as

$$\dot{z}_i(t) = A_{ii} z_i(t) + \sum_{j \in \mathcal{N}_i^{\mathrm{u}}} A_{ij} z_j(t) + B_i u_i(t),$$

where $A_{il} = \partial f_i / \partial z_l(0,0)$ and $B_i = \partial f_i / \partial u_i(0,0)$. As in many RHC formulations, a feedback controller for which the closed-loop system is asymptotically stabilized inside a neighborhood of the origin will be utilized. There exist methods for constructing dynamic and static feedback controllers, as done by Corfmat and Morse in [3], to achieve stabilization while respecting the decentralized information constraints. The analysis here is greatly facilitated if, for every $i \in \mathcal{V}$, stabilization is possible with the static feedback $u_i = K_i z_i$, instead of a feedback that relies on components of z_{-i}, motivating the following assumption.

Assumption 2: For every agent $i \in \mathcal{V}$, there exists a decoupled static feedback $u_i = K_i z_i$ such that $A_{di} \triangleq A_{ii} + B_i K_i$ is Hurwitz, and the closed-loop linear system $\dot{z} = A_{\mathrm{c}} z$ is asymptotically stable, where $A_{\mathrm{c}} \triangleq [f_z(0,0) + f_u(0,0)K]$ and $K = \mathrm{diag}(K_1, ..., K_{N_a})$.

The decoupled linear feedbacks above are referred to as *terminal controllers*. Associated with the closed-loop linearization, denote the block-diagonal Hurwitz matrix $A_{\mathrm{d}} = \mathrm{diag}(A_{d1}, ..., A_{dN_a})$ and the off-diagonal matrix $A_{\mathrm{o}} = A_{\mathrm{c}} - A_{\mathrm{d}}$. Assumption 2 inherently presumes decoupled stabilizability and that the coupling between subsystems in the linearization is sufficiently weak, as discussed and quantified in the survey paper [12]. The terminal controllers will be employed in a prescribed neighborhood of the origin. Thus, the coupling between subsystems must be sufficiently weak in the prescribed neighborhood. The degree of weakness required here will be stated as a mathematical condition in the next section. Since the stabilizing controllers constructed in [3] do not require the assumption of weak coupling between subsystems, a future objective is to admit such terminal controllers.

III. Distributed Receding Horizon Control

In this section, N_a separate optimal control problems are defined and the distributed RHC algorithm. In every distributed optimal control problem, the same constant prediction horizon $T \in (0, \infty)$ and constant update period $\delta \in (0, T]$ are used. In practice, the update period $\delta \in (0, T]$ is typically the sample interval. By the distributed implementation presented here, additional conditions on δ are required, as quantified in the next section. Denote the update time $t_k = t_0 + \delta k$, where $k \in \mathbb{N} = \{0, 1, 2, ...\}$. In the following implementation, every distributed RHC law is updated *globally synchronously*, i.e., at the same instant of time t_k for the k^{th}-update.

At each update, every agent optimizes only for its own predicted open-loop control, given its current state. Since the dynamics of each agent i depend on states z_{-i}, that agent will presume some trajectories for z_{-i} over each prediction horizon. To that end, prior to each update, each agent i *receives* an *assumed* state trajectory $\hat{z}_j$ from each upstream neighbor $j \in \mathcal{N}_i^{\text{u}}$. Likewise, agent i transmits an assumed state trajectory $\hat{z}_i$ to every downstream neighbor $j \in \mathcal{N}_i^{\text{d}}$, prior to each update. By design, then, the assumed state trajectory for any agent is *the same* in the distributed optimal control problem of every downstream neighbor. Since the models are used with assumed trajectories for upstream neighbors, there will be a discrepancy, over each optimization time window, between the predicted open-loop trajectory and the actual trajectory that results from every agent applying the predicted control. This discrepancy is identified by using the following notation. Recall that $z_i(t)$ and $u_i(t)$ are the actual state and control, respectively, for each agent $i \in \mathcal{V}$ at any time $t \geq t_0$. Associated with update time t_k, the trajectories for each agent $i \in \mathcal{V}$ are denoted

$z_i^p(t;t_k)$ – the predicted state trajectory,
$\hat{z}_i(t;t_k)$ – the assumed state trajectory,
$u_i^p(t;t_k)$ – the predicted control trajectory,

where $t \in [t_k, t_k+T]$. Consistent with the ordering of z_{-i}, let $\hat{z}_{-i}(\cdot;t_k)$ be the assumed open-loop state trajectories of the upstream neighbors of i. For any agent $i \in \mathcal{V}$, then, the predicted state trajectory satisfies

$$\dot{z}_i^p(t;t_k) = f_i(z_i^p(t;t_k), \hat{z}_{-i}(t;t_k), u_i^p(t;t_k)), \tag{3}$$

for all $t \in [t_k, t_k+T]$, given $z_i^p(t_k;t_k) = z_i(t_k)$ and $\hat{z}_{-i}(\cdot;t_k)$. The actual state trajectories, on the other hand, satisfy

$$\dot{z}(t) = f(z(t), u^p(t;t_k)), \tag{4}$$

for time $t \in [t_k, t_{k+1}]$, for any $k \in \mathbb{N}$, given $z(t_k)$, and where $u^p = (u_1^p, ..., u_{N_a}^p)$. So, while the actual and predicted state trajectories do have the same initial condition, they typically diverge over each window $[t_k, t_{k+1}]$, and $z_i^p(t_{k+1};t_k) \neq z_i(t_{k+1})$ in general for any $i \in \mathcal{V}$. Stability is to be guaranteed for *the closed-loop system*, represented by equation (4), which is defined for all time $t \geq t_0$ by following the distributed RHC algorithm. For each agent i, an assumed control trajectory $\hat{u}_i(\cdot;t_k)$ is also utilized, and $\hat{z}_i(\cdot;t_k)$ and $\hat{u}_i(\cdot;t_k)$ are defined in the algorithm.

The cost function $J_i(z_i(t_k), u_i^p(\cdot;t_k))$ for any agent $i \in \mathcal{V}$ at update time t_k is given by

$$\int_{t_k}^{t_k+T} \|z_i^p(s;t_k)\|_{Q_i}^2 + \|u_i^p(s;t_k))\|_{R_i}^2 \mathrm{d}s + \|z_i^p(t_k+T;t_k)\|_{P_i}^2 \tag{5}$$

where $Q_i = Q_i^T > 0$ and $R_i = R_i^T > 0$. The terminal cost matrix $P_i = P_i^T > 0$ satisfies the Lyapunov equation $P_i A_{di} + A_{di}^T P_i = -4\widehat{Q}_i$, where $\widehat{Q}_i = Q_i + K_i^T R_i K_i$. Now, let $P = \text{diag}(P_1, ..., P_{N_a})$ and $\widehat{Q} = \text{diag}(\widehat{Q}_1, ..., \widehat{Q}_{N_a})$, and so $PA_{\text{d}} + A_{\text{d}}^T P = -4\widehat{Q}$ with $\widehat{Q} > 0$. The following assumption quantifies the presumed degree of weakness of coupling between neighboring subsystems in the linearization.

Assumption 3: $PA_{\text{o}} + A_{\text{o}}^T P \leq 2\widehat{Q}$.

Lemma 1: Suppose that Assumptions 1–3 hold. There exists a positive constant $\varepsilon \in (0, \infty)$ such that the set

$$\Omega_\varepsilon \triangleq \left\{ z \in \mathbb{R}^{nN_a} \mid \|z\|_P \leq \varepsilon \right\},$$

is a positively invariant region of attraction for both the closed-loop linearization $\dot{z}(t) = A_{\text{c}} z(t)$ and the closed-loop nonlinear system $\dot{z}(t) = f(z(t), Kz(t))$. Additionally, $Kz \in \mathcal{U}^{N_a}$ for all $z \in \Omega_\varepsilon$.
The proof follows closely along the lines of the logic given in Section II of [10] and is omitted for space reasons. For each $i \in \mathcal{V}$, define the set

$$\Omega_i(\varepsilon) \triangleq \left\{ z_i \in \mathbb{R}^n \mid \|z_i\|_{P_i} \leq \varepsilon/\sqrt{N_a} \right\}.$$

Then, $z(t) \in \Omega_1(\varepsilon) \times \cdots \times \Omega_{N_a}(\varepsilon)$ implies $z(t) \in \Omega_\varepsilon$. Consequently, from Lemma 1, if at some time $t' \geq t_0$, $z_i(t') \in \Omega_i(\varepsilon)$ for every $i \in \mathcal{V}$, stabilization is achieved if every agent employs their decoupled terminal controller $K_i z_i(t)$ for all time $t \geq t'$. The objective of the RHC law is to drive each agent i to the set $\Omega_i(\varepsilon)$. Once all agents have reached these sets, they switch to their decoupled controllers for stabilization. The collection of distributed optimal control problems is now defined.

Problem 1: For each agent $i \in \mathcal{V}$ and at any update time t_k, $k \in \mathbb{N}$, given $\{z_i(t_k),\ J_i(z_i(t_k), \hat{u}_i(\cdot;t_k)),\ \hat{u}_i(\tau;t_k),\ \hat{z}_{-i}(\tau;t_k),\ \forall \tau \in [t_k, t_k+T]\}$, find $u_i^p(\cdot;t_k)$ that satisfies

$$J_i(z_i(t_k), u_i^p(\cdot;t_k)) \leq J_i(z_i(t_k), \hat{u}_i(\cdot;t_k)), \tag{6}$$

subject to equation (3), $u_i^p(\tau;t_k) \in \mathcal{U}$, the *control compatibility constraint*

$$\|u_i^p(\tau;t_k) - \hat{u}_i(\tau;t_k)\|_{W_i} \leq c_1, \tag{7}$$

$\tau \in [t_k, t_k+T]$, and the terminal state constraint $z_i^p(t_k + T;t_k) \in \Omega_i(\varepsilon/2)$, given weighting $W_i \geq 0$, positive constant c_1 and ε satisfies the conditions in Lemma 1. ■
While the cost $J_i(z_i(t_k), u_i^p(\cdot;t_k))$ is defined by equation (5), the cost $J_i(z_i(t_k), \hat{u}_i(\cdot;t_k))$ will be defined later by equation (8), after the distributed algorithm is stated. Before stating the algorithm, an assumption is made to facilitate initialization.

Assumption 4: Given $z(t_0)$ at initial time t_0, there exists a feasible control $u_i^p(\tau;t_0) \in \mathcal{U}$, $\tau \in [t_0, t_0+T]$, for each agent $i \in \mathcal{V}$, such that the solution to the full system $\dot{z}(\tau) = f(z(\tau), u^p(\tau;t_0))$, denoted $z^p(\cdot;t_0)$, satisfies $z_i^p(t_0+T;t_0) \in \Omega_i(\varepsilon/2)$ and results in a bounded cost $J_i(z_i(t_0), u_i^p(\cdot;t_0))$ for every $i \in \mathcal{V}$. Moreover, each agent $i \in \mathcal{V}$ has access to $(z_i^p(\cdot;t_0), u_i^p(\cdot;t_0))$.
Let $Z \subset \mathbb{R}^{nN_a}$ denote the set of initial states for which there exists a control satisfying the conditions in Assumption 4. Of course, Assumption 4 bypasses the difficult task of actually constructing an initially feasible solution in a distributed way, a task that is left for future work. The control algorithm is now stated.

Algorithm 1: At time t_0 with $z(t_0) \in Z$, the dual-mode *Distributed Receding Horizon Control* law for any agent $i \in \mathcal{V}$ is as follows:

Data: $z(t_0)$, $u_i^p(\cdot; t_0)$, $T \in (0, \infty)$, $\delta \in (0, T]$.

Initialization: At time t_0, if $z(t_0) \in \Omega_\varepsilon$, then apply the terminal controller $u_i(t) = K_i z_i(t)$, for all $t \geq t_0$. Else:

Controller: **A**) Over any interval $[t_k, t_{k+1})$, $k \in \mathbb{N}$:

1) Apply $u_i^p(\tau; t_k)$, $\tau \in [t_k, t_{k+1})$.
2) Compute

$$\hat{z}_i(\tau; t_{k+1}) = \begin{cases} z_i^p(\tau; t_k), & \tau \in [t_{k+1}, t_k + T) \\ z_i^K(\tau), & \tau \in [t_k + T, t_{k+1} + T] \end{cases}$$

where z_i^K is the solution to $\dot{z}_i^K(\tau) = A_{di} z_i^K(\tau)$ with initial condition $z_i^K(t_k + T) = z_i^p(t_k + T; t_k)$.
3) Transmit $\hat{z}_i(\cdot; t_{k+1})$ to every downstream neighbor $l \in \mathcal{N}_i^{\rm d}$, and receive $\hat{z}_j(\cdot; t_{k+1})$ from every upstream neighbor $j \in \mathcal{N}_i^{\rm u}$.

B) At any time t_{k+1}, $k \in \mathbb{N}$:

1) Obtain $z(t_{k+1})$. If $z(t_{k+1}) \in \Omega_\varepsilon$, apply the terminal controller $u_i(t) = K_i z_i(t)$, for all $t \geq t_{k+1}$. Else:
2) Compute the state trajectory $\bar{z}_i(\cdot; t_{k+1})$ and assumed control trajectory $\hat{u}_i(\cdot; t_{k+1})$ according to

$$\dot{\bar{z}}_i(\tau; t_{k+1}) = f_i(\bar{z}_i(\tau; t_{k+1}), \hat{z}_{-i}(\tau; t_{k+1}), \hat{u}_i(\tau; t_k)),$$

$$\hat{u}_i(\tau; t_{k+1}) = \begin{cases} u_i^p(\tau; t_k), & \tau \in [t_{k+1}, t_k + T) \\ K_i \bar{z}_i(\tau; t_{k+1}), & \tau \in [t_k + T, t_{k+1} + T] \end{cases}$$

for all $\tau \in [t_{k+1}, t_{k+1} + T]$, with initial condition $\bar{z}_i(t_{k+1}; t_{k+1}) = z_i(t_{k+1})$.
3) Compute the cost $J_i(z_i(t_{k+1}), \hat{u}_i(\cdot; t_{k+1}))$ as

$$\int_{t_{k+1}}^{t_{k+1}+T} \|\bar{z}_i(s; t_{k+1})\|_{Q_i}^2 + \|\hat{u}_i(s; t_{k+1}))\|_{R_i}^2 \mathrm{d}s + \|\bar{z}_i(t_{k+1} + T; t_{k+1})\|_{P_i}^2. \quad (8)$$

4) Solve Problem 1 for agent i, yielding $u_i^p(\cdot; t_{k+1})$. ■

Algorithm 1 presumes that the every agent can obtain $z(t_k)$ at every $k \in \mathbb{N}$. This can be done using a single centralized node, or in a distributed way using a consensus protocol as discussed in [4]. By construction, each assumed state trajectory $\hat{z}_i$ is the remainder of the previously predicted trajectory, concatenated with the closed-loop linearization response that ignores coupling. Each assumed control trajectory $\hat{u}_i$ is the remainder of the previously predicted trajectory, concatenated with the linear control applied to the nonlinear model and based on the decoupled linear responses for each upstream neighbor. Observe that, for any $i \in \mathcal{V}$ and $k \geq 2$, $\hat{z}_i(t_k; t_k) = z_i^p(t_k; t_{k-1}) \neq \bar{z}_i(t_k; t_k) = z_i(t_k)$.

IV. Analysis

In this section, feasibility and stability are analyzed. A desirable property of the implementation is that initial feasibility implies subsequent feasibility. In particular, it will be shown in Theorem 1 that a feasible solution to Problem 1 for any $i \in \mathcal{V}$ and at any time t_k, $k \geq 1$, is $u_i^p(\cdot; t_k) = \hat{u}_i(\cdot; t_k)$. First, three lemmas are given for use in the proof of Theorem 1. Specifically, terminal constraint feasibility will be shown by combining Lemmas 2 and 3, while control constraint feasibility is shown in Lemma 4. Define the function $\Theta(t; t_k) \triangleq \sum_{i\in\mathcal{V}} \|\bar{z}_i(t; t_k) - \hat{z}_i(t; t_k)\|_{P_i}$, for all $t \in [t_k, t_k + T]$, and the following positive constants: $\alpha_m = \max_{i\in\mathcal{V}}[\alpha_i]$, $\beta_m = \max_{i\in\mathcal{V}}[\sum_{j\in\mathcal{N}_i^{\rm d}} \beta_j^i]$, $\gamma_m = \max_{i\in\mathcal{V}}[\gamma_i]$, $\gamma_W = \gamma_m \lambda_{\max}^{1/2}(K^T W K)/\lambda_{\min}^{1/2}(P)$, where $W = \mathrm{diag}(W_1, ..., W_{N_a})$.

Lemma 2: Suppose that Assumptions 1–4 hold and $z(t_0) \in Z$. In the application of Algorithm 1, suppose that Problem 1 has a solution at each update $l = 1, ..., k$, for any given $k \geq 1$. Then, $\Theta(t; t_{k+1}) \leq \varepsilon/(4\sqrt{N_a})$, $\forall t \in [t_{k+1}, t_{k+1} + T]$, provided the following parametric conditions hold:

$$\delta \cdot \max\left\{ \frac{16\lambda_{\min}(\widehat{Q})}{N_a^{3/2}\lambda_{\max}(P)}, (\alpha_m + \gamma_W) \right\} \leq 1/2, \quad (9)$$

$$\beta_m \cdot \max\{8\delta N_a, T e^{\alpha_m(T+\delta)+\beta_m\delta}\} \leq 1/4, \quad (10)$$

$$c_1 \leq \frac{\varepsilon(1 + \delta\gamma_W)}{32 T \gamma_m N_a^{3/2} e^{\alpha_m(T-\delta)}}. \quad (11)$$

Proof. Define the function $y(t; t_l) \triangleq \sum_{i\in\mathcal{V}} \|z_i^p(t; t_l) - \hat{z}_i(t; t_l)\|_{P_i}$, $t \in [t_l, t_l + T]$, $l = 1, ..., k$. This function and $\Theta(t; t_{k+1})$ are well defined by assuming the existence of a solution to Problem 1 at each update $l = 1, ..., k$. Using the Lipschitz bounds stated in Assumption 1, the triangle and Gronwall-Bellman inequalities, it is straightforward to show that $\Theta(t; t_{k+1}) \leq \Gamma_k$ for all $t \in [t_{k+1}, t_k + T]$, where

$$\Gamma_0 \triangleq 0, \quad \Gamma_l \triangleq \beta_m e^{\alpha_m T + \beta_m \delta} \int_{t_l}^{t_l+T} y(s; t_l) \mathrm{d}s,$$

for $l = 1, ..., k$. Let $w(t; t_l) \triangleq \sum_{i\in\mathcal{V}} \|u_i^p(t; t_l) - \hat{u}_i(t; t_l)\|_{W_i}$, $t \in [t_l, t_l + T]$, for $l = 1, ..., k$. In the same fashion, it is straightforward to show that

$$y(t; t_l) \leq e^{\alpha_m(t-t_l)} \Big[\beta_m e^{\delta(\alpha_m+\beta_m)} \int_{t_{l-1}}^{t_{l-1}+T} y(s; t_{l-1}) \mathrm{d}s + \gamma_m \int_{t_l}^{t_{l-1}+T} w(s; t_l) \mathrm{d}s \Big],$$

for all $t \in [t_l, t_{l-1} + T]$ and

$$y(t; t_l) \leq y(t_{l-1} + T; t_l) + \int_{t_{l-1}+T}^{t} \sum_{i\in\mathcal{V}} \|f_i(z_i^p(s; t_l), \hat{z}_{-i}(s; t_l), u_i^p(s; t_l) - A_{di}\hat{z}_i(s; t_l) \pm f_i(\hat{z}_i(s; t_l), 0, K_i \hat{z}_i(s; t_l))\|_{P_i} \mathrm{d}s,$$

for all $t \in [t_{l-1} + T, t_l + T]$, $l = 1, ..., k$. To simplify the integrand, analysis in the proof of Lemma 1, the Lipschitz bound and triangle inequality are combined to yield

$$y(t; t_l) \leq y(t_{l-1} + T; t_l) + \int_{t_{l-1}+T}^{t} \Big[\alpha_m y(s; t_l) + \gamma_m w(s; t_l) + \beta_m \sum_{i\in\mathcal{V}} \|\hat{z}_i(s; t_l)\|_{P_i} + \frac{\lambda_{\min}(\widehat{Q})}{N_a^2 \lambda_{\max}(P)} \|\hat{z}(s; t_l)\|_P + \gamma_m \sum_{i\in\mathcal{V}} \|\hat{u}_i(s; t_l) - K_i \hat{z}_i(s; t_l)\|_{W_i} \Big] \mathrm{d}s.$$

Over the domain $[t_{l-1}+T, t_l+T]$, $\|\hat{z}_i(s;t_l)\|_{P_i} \le \varepsilon/(2\sqrt{N_a})$ for every $i \in \mathcal{V}$, and so $\|\hat{z}(s;t_l)\|_P \le \varepsilon/2$. Also, the last term in the integrand is bounded by $\gamma_W \Theta(s;t_l)$. Now, using Gronwall-Bellman and the previously obtained bound on $y(t_{l-1}+T;t_l)$, it follows that for all $t \in [t_l, t_l+T]$,

$$y(t;t_l) \le \beta_m e^{\alpha_m(T+\delta)+\beta_m\delta} \int_{t_{l-1}}^{t_{l-1}+T} y(s;t_{l-1})\mathrm{d}s + e^{\alpha_m\delta}X$$
$$+ \gamma_m e^{\alpha_m T} \int_{t_l}^{t_l+T} w(s;t_l)\mathrm{d}s + \gamma_W e^{\alpha_m\delta} \int_{t_{l-1}+T}^{t_l+T} \Theta(s;t_l)\mathrm{d}s,$$

where $X = \delta\varepsilon\{\beta_m\sqrt{N_a} + \lambda_{\min}(\widehat{Q})/(N_a^2\lambda_{\max}(P))\}/2$. The bound on $\Theta(t;t_{k+1})$ over the time domain $t \in [t_k+T, t_{k+1}+T]$ is likewise given by

$$\Theta(t;t_{k+1}) \le \Theta(t_k+T;t_{k+1}) + \int_{t_k+T}^{t} \Big[\alpha_m\Theta(t;t_{k+1})$$
$$+ \beta_m \sum_{i\in\mathcal{V}} \|\hat{z}_i(s;t_{k+1})\|_{P_i} + \frac{\lambda_{\min}(\widehat{Q})}{N_a^2\lambda_{\max}(P)}\|\hat{z}(s;t_{k+1})\|_P$$
$$\gamma_m \sum_{i\in\mathcal{V}} \|\hat{u}_i(s;t_{k+1}) - K_i\hat{z}_i(s;t_{k+1})\|_{W_i}\Big]\mathrm{d}s.$$

Now, using Gronwall-Bellman and the previously obtained bound on $\Theta(t_k+T;t_{k+1})$, it follows that $\Theta(t;t_{k+1}) \le Y_{k+1}$ for all $t \in [t_{k+1}, t_{k+1}+T]$, where $Y_{k+1} \triangleq e^{\delta(\alpha_m+\gamma_W)}(X+\Gamma_k)$. Using the bound for $y(t;t_l)$ and the definition for Γ_l yields

$$\Gamma_l \le \beta_m T e^{\alpha_m(T+\delta)+\beta_m\delta}\Big[\Gamma_{l-1} + X \tag{12}$$
$$+ \gamma_m e^{\alpha_m(T-\delta)}N_a c_1 + \delta\gamma_W e^{\delta(\alpha_m+\gamma_W)}(X+\Gamma_{l-1})\Big]$$
$$\triangleq \rho_1(\Gamma_{l-1}+\rho_2), \quad l = 1, ..., k, \tag{13}$$

Now, if $\rho_1 \le 1/2$ and $\rho_2 \le \varepsilon/(16\sqrt{N_a})$, then $\Gamma_l \le \varepsilon/(16\sqrt{N_a})$ for all $l = 1, ..., k$, and $X \le \varepsilon/(16\sqrt{N_a})$. If, in addition, $e^{\delta(\alpha_m+\gamma_W)} \le 2$, then from the definition of Y_{k+1}, it follows that $Y_{k+1} \le \varepsilon/(4\sqrt{N_a})$. Finally, since $\Theta(t;t_{k+1}) \le Y_{k+1}$ for all $t \in [t_{k+1}, t_{k+1}+T]$, the result of the lemma follows, provided:

$$\rho_1 \le 1/2,\ \rho_2 \le \varepsilon/(16\sqrt{N_a}),\ \text{and } e^{\delta(\alpha_m+\gamma_W)} \le 2. \tag{14}$$

Conditions (9)–(11) are now shown to imply that (14) holds. Condition (9) implies $e^{\delta(\alpha_m+\gamma_W)} \le 2$ and $\delta\gamma_W e^{\delta(\alpha_m+\gamma_W)} \le 1$, and combining with (10) implies $\rho_1 \le 1/2$. With (11),

$$\gamma_m T N_a c_1 e^{\alpha_m(T-\delta)} / \left\{1 + \delta\gamma_W e^{\delta(\alpha_m+\gamma_W)}\right\} \le \varepsilon/(32\sqrt{N_a}).$$

Also, (9) and (10) imply that $X \le \varepsilon/(32\sqrt{N_a})$. Combining these last two bounds implies that $\rho_2 \le \varepsilon/(16\sqrt{N_a})$, concluding the proof. ■

The conservative approach taken in the proof results in sufficient conditions (9)–(11) that are likewise conservative. Condition (10) requires that the degree of dynamic coupling (parameterized by β_m) be sufficiently weak. In contrast, conditions (11) and (9) are design constraints. To ensure feasibility, the requirements that agents can not deviate too far from the behavior neighbors assume for them (from (11),(7)), and that the sample rate not be too large (from (9)) are intuitive. We now proceed with the second lemma used to ensure terminal constraint feasibility by the assumed control.

Lemma 3: Suppose that Assumptions 1–4 hold and $z(t_0) \in Z$. In the application of Algorithm 1, suppose that Problem 1 has a solution at each update $l = 1, ..., k$, for any given $k \ge 1$. Then, for every $i \in \mathcal{V}$, $\|\hat{z}_i(t_{k+1}+T;t_{k+1})\|_{P_i} \le \varepsilon/(4\sqrt{N_a})$, provided the following parametric condition holds:

$$\delta\lambda_{\min}(\widehat{Q})/\lambda_{\max}(P) \ge \ln(4N_a)/4. \tag{15}$$

Proof. By construction, for every $i \in \mathcal{V}$, it follows from the terminal constraint that $\|\hat{z}_i(t_k+T;t_{k+1})\|_{P_i} = \|z_i^p(t_k+T;t_k)\|_{P_i} \le \varepsilon/(2\sqrt{N_a})$, and so $\hat{z}(t_k+T;t_{k+1}) \in \Omega_{\varepsilon/2}$. With the Lyapunov function $V(\hat{z}(t)) = \|\hat{z}(t;t_{k+1})\|_P^2$ for $t \in [t_k+T, t_{k+1}+T]$, it follows that $\dot{V}(\hat{z}(t)) \le -4\|\hat{z}(t;t_{k+1})\|_{\widehat{Q}}^2 \le -4[\lambda_{\min}(\widehat{Q})/\lambda_{\max}(P)]V(\hat{z}(t))$. Consequently, $V(\hat{z}(t)) \le \exp[-4(t-(t_k+T))\lambda_{\min}(\widehat{Q})/\lambda_{\max}(P)]V(\hat{z}(t_k+T))$. A sufficient condition for $\|\hat{z}_i(t_{k+1}+T;t_{k+1})\|_{P_i} \le \varepsilon/(4\sqrt{N_a})$ for every $i \in \mathcal{V}$ is that $\|\hat{z}(t_{k+1}+T;t_{k+1})\|_P \le \varepsilon/(4\sqrt{N_a})$. Using the bound on the Lyapunov function, the sufficient condition requires that $e^{-4\delta\lambda_{\min}(\widehat{Q})/\lambda_{\max}(P)}V(\hat{z}(t_k+T)) \le \varepsilon/(4\sqrt{N_a})$, or, more conservatively, that $e^{-4\delta\lambda_{\min}(\widehat{Q})/\lambda_{\max}(P)}\varepsilon^2/4 \le \varepsilon^2/(16N_a)$, which is equivalent to (15), concluding the proof. ■

Condition (15) suggests a (conservative) minimum amount of time δ required to steer each $z_i^p(t_k+T;t_k)$ from $\Omega_i(\varepsilon/2)$ to $\Omega_i(\varepsilon/4)$, using the decoupled terminal controllers. Since condition (9) places an upper bound on δ, it must be ensured these two conditions are compatible. Rewriting the conditions, it is required that

$$\ln(4N_a)/4 \le \delta\lambda_{\min}(\widehat{Q})/\lambda_{\max}(P) \le N_a^{3/2}/32. \tag{16}$$

Provided $N_a \ge 10$, there always exists a feasible $\delta\lambda_{\min}(\widehat{Q})/\lambda_{\max}(P)$ that satisfies (16). It is now shown that the assumed controls satisfy the control constraints when the conditions of Lemma 2 are satisfied.

Lemma 4: Suppose that Assumptions 1–4 hold, $z(t_0) \in Z$ and conditions (9)–(11) are satisfied. In the application of Algorithm 1, suppose that Problem 1 has a solution at each update $l = 1, ..., k$, for any given $k \ge 1$. Then, for every $i \in \mathcal{V}$, $\hat{u}_i(\tau;t_{k+1}) \in \mathcal{U}$ for all $\tau \in [t_{k+1}, t_{k+1}+T]$.

Proof. Since $\hat{u}_i(t;t_{k+1}) = u_i^p(t;t_k)$ for all $t \in [t_{k+1}, t_k+T]$, it need only be shown that the remainder of $\hat{u}_i$ is in $\mathcal{U}$. A sufficient condition for this is if $\bar{z}_i(t;t_{k+1}) \in \Omega_i(\varepsilon)$ for all $t \in [t_k+T, t_{k+1}+T]$, since ε is chosen to satisfy the conditions of Lemma 1 and, consequently, $K_i z_i \in \mathcal{U}$ for all $i \in \mathcal{V}$ when $z \in \Omega_\varepsilon$. From Lemma 2, $\|\bar{z}_i(t;t_{k+1}) - \hat{z}_i(t;t_{k+1})\|_{P_i} \le \varepsilon/(4\sqrt{N_a})$ for all $t \in [t_k+T, t_{k+1}+T]$. By construction, $\|\hat{z}_i(t_k+T;t_{k+1})\|_{P_i} \le \varepsilon/(2\sqrt{N_a})$. With the Lyapunov function $V_i(\hat{z}_i(t)) = \|\hat{z}_i(t;t_{k+1})\|_{P_i}^2$, it follows that $\dot{V}(\hat{z}_i(t)) \le -4[\lambda_{\min}(\widehat{Q}_i)/\lambda_{\max}(P_i)]V_i(\hat{z}_i(t))$, $\forall t \in [t_k+T, t_{k+1}+T]$, and so $\|\hat{z}_i(t;t_{k+1})\|_{P_i} \le \varepsilon/(2\sqrt{N_a})$ for all $t \in [t_k+T, t_{k+1}+T]$. Using the triangle inequality, $\|\bar{z}_i(t;t_{k+1})\|_{P_i} = \|\bar{z}_i(t;t_{k+1}) \pm \hat{z}_i(t;t_{k+1})\|_{P_i} \le \|\bar{z}_i(t;t_{k+1}) - \hat{z}_i(t;t_{k+1})\|_{P_i} + \|\hat{z}_i(t;t_{k+1})\|_{P_i} \le \varepsilon/(4\sqrt{N_a}) + \varepsilon/(2\sqrt{N_a}) < \varepsilon/\sqrt{N_a}$, and so $\bar{z}_i(t;t_{k+1}) \in \Omega_i(\varepsilon)$, $\forall t \in$

$[t_k+T, t_{k+1}+T]$ for every $i \in \mathcal{V}$, concluding the proof. ■

The first main theorem of the paper is now stated.

Theorem 1: Suppose that Assumptions 1–4 hold, $z(t_0) \in Z$ and conditions (9)–(15) are satisfied. Then, for every agent $i \in \mathcal{V}$, the assumed control $\hat{u}_i(\cdot; t_k)$ is a feasible solution to Problem 1 and at every update $k \geq 1$.

Proof. The proof follows by induction. First, the $k = 1$ case. The trajectory $\hat{u}_i(\cdot; t_1)$ trivially satisfies the cost improvement constraint (6) and the control compatibility constraint (7). According to Algorithm 1, $\bar{z}_i(\cdot; t_1)$ is the corresponding state trajectory that satisfies the dynamic equation. Now, observe that $\hat{z}_i(t_1; t_1) = z_i^p(t_1; t_0) = \bar{z}_i(t_1; t_1) = z_i(t_1)$ for every $i \in \mathcal{V}$. Additionally, $\bar{z}_i(t; t_1) = z_i^p(t; t_0)$ for all $t \in [t_1, t_0+T]$, and so $\bar{z}_i(t_0+T; t_1) \in \Omega_i(\varepsilon/2)$. By the invariance properties of the terminal controller and the conditions in Lemma 1, it follows that the terminal state and control constraints are also satisfied, concluding the $k = 1$ case. Now, the induction step. By assumption, suppose $u_i^p(\cdot; t_l) = \hat{u}_i(\cdot; t_l)$ is a feasible solution for $l = 1, ..., k$. It must be shown that $\hat{u}_i(\cdot; t_{k+1})$ is a feasible solution at update $k+1$. As before, the cost improvement constraint (6) and the control compatibility constraint (7) are trivially satisfied, and $\bar{z}_i(\cdot; t_{k+1})$ is the corresponding state trajectory that satisfies the dynamic equation. Since there is a solution for updates $l = 1, ..., k$, Lemmas 2–4 can be invoked. Lemma 4 guarantees control constraint feasibility. The terminal constraint requires $\bar{z}_i(t_k+T; t_k) \in \Omega_i(\varepsilon/2)$, for each $i \in \mathcal{V}$ and every $k \geq 1$. From Lemma 2, $\|\bar{z}_i(t_k+T; t_k) - \hat{z}_i(t_k+T; t_k)\|_{P_i} \leq \varepsilon/(4\sqrt{N_a})$, and Lemma 3 guarantees that $\|\hat{z}_i(t_k+T; t_k)\|_{P_i} \leq \varepsilon/(4\sqrt{N_a})$. Combining these two bounds, and with $\|\bar{z}_i(t; t_k) \pm \hat{z}_i(t; t_k)\|_{P_i} \leq \|\bar{z}_i(t; t_k) - \hat{z}_i(t; t_k)\|_{P_i} + \|\hat{z}_i(t; t_k)\|_{P_i}$, it follows that $\|\bar{z}_i(t_k+T; t_k)\|_{P_i} \leq \varepsilon/(2\sqrt{N_a})$, concluding the proof. ■

Now, stability of the closed-loop system (4) is analyzed. At any time t_k, $k \in \mathbb{N}$, the sum of the distributed cost functions is denoted $J_\Sigma(t_k) = \sum_{i=1}^{N_a} J_i(z_i(t_k), u^p(\cdot; t_k))$.

Assumption 5: For every agent $i \in \mathcal{V}$, there exists a common positive constant $\rho_{\max} \in (0, \infty)$ satisfying $\|z_i^p(t; t_k)\|_{P_i} \leq \rho_{\max}/\sqrt{N_a}$, $\hat{z}_i(t; t_k) \leq \rho_{\max}/\sqrt{N_a}$ and $\bar{z}_i(t; t_k) \leq \rho_{\max}/\sqrt{N_a}$, for all $t \in [t_k, t_k+T]$ and any $k \geq 1$. Boundedness assumptions of this form are standard in convergence analysis of optimization algorithms.

Theorem 2: Suppose that Assumptions 1–5 hold, $z(t_0) \in Z$ and conditions (9)–(15) are satisfied. Then, by application of Algorithm 1, the closed-loop system (4) is asymptotically stabilized to the origin, provided the following parametric conditions are satisfied:

$$\delta \lambda_{\min}(Q)/\lambda_{\max}(P) \geq \ln(4N_a)/4 \tag{17}$$

$$N_a \geq \frac{(T-\delta)(8\rho_{\max}+\varepsilon)\lambda_{\max}(\widehat{Q})}{135\varepsilon\lambda_{\min}(P)}. \tag{18}$$

Proof. If $z(t_k) \in \Omega_\varepsilon$ for any $k \geq 0$, the terminal controllers take over and stabilize the system to the origin. Therefore, it remains to show that if $z(t_0) \in Z \setminus \Omega_\varepsilon$, then by application of Algorithm 1, the closed-loop system (4) is driven to the set Ω_ε in finite time. This is shown by demonstrating that if $z(t_k), z(t_{k+1}) \in Z \setminus \Omega_\varepsilon$, then $J_\Sigma(t_{k+1}) - J_\Sigma(t_k) \leq -C$ for some positive constant C. From this inequality, there exists a finite integer $l > 1$ such that $z(t_l) \in \Omega_\varepsilon$, by contradiction. If this were not the case, the inequality implies $J_\Sigma(t_k) \to -\infty$ as $k \to \infty$. However, $J_\Sigma(t_k) \geq 0$; therefore, there exists a finite integer $l \geq 1$ such that $z(t_l) \in \Omega_\varepsilon$.

It remains to show that if $z(t_k), z(t_{k+1}) \in Z \setminus \Omega_\varepsilon$, then $J_\Sigma(t_{k+1}) - J_\Sigma(t_k) \leq -C$ for some positive constant C. From (5), $J_\Sigma(t_k)$ is

$$\int_{t_k}^{t_k+T} \|z^p(s; t_k)\|_Q^2 + \|u^p(s; t_k))\|_R^2 \mathrm{d}s + \|z^p(t_k+T; t_k)\|_P^2.$$

From (6) and (8),

$$J_\Sigma(t_{k+1}) \leq \int_{t_{k+1}}^{t_{k+1}+T} \|\bar{z}(s; t_{k+1})\|_Q^2 + \|\hat{u}(s; t_{k+1}))\|_R^2 \mathrm{d}s + \|\bar{z}(t_{k+1}+T; t_{k+1})\|_P^2.$$

Subtracting the two equations implies

$$\begin{aligned} J_\Sigma(t_{k+1}) - J_\Sigma(t_k) &\leq -\int_{t_k}^{t_{k+1}} \|z^p(s; t_k)\|_Q^2 \mathrm{d}s \\ &+ \int_{t_{k+1}}^{t_k+T} \|\bar{z}(s; t_{k+1})\|_Q^2 - \|\hat{z}(s; t_{k+1})\|_Q^2 \mathrm{d}s - \|z^p(t_k+T; t_k)\|_P^2 \\ &+ \int_{t_k+T}^{t_{k+1}+T} \|\bar{z}(s; t_{k+1})\|_Q^2 \mathrm{d}s + \|\bar{z}(t_{k+1}+T; t_{k+1})\|_P^2. \end{aligned}$$

The remainder of the proof is concerned with bounding each of the terms on the right of the inequality. It can be shown that if $z(t^1) \in \Omega_{7\varepsilon/8}$ at any time $t^1 \in (t_k, t_{k+1}]$, then $\max_{t \in (t_k, t_{k+1}]} \|z(t)\|_P \leq \varepsilon$. Consequently, $z(t_{k+1}) \in \Omega_\varepsilon$. Since $z(t_k), z(t_{k+1}) \in Z \setminus \Omega_\varepsilon$ is assumed, it must be that $\|z(t)\|_P > 7\varepsilon/8$ for all time $t \in [t_k, t_{k+1}]$. From the triangle inequality, $\|z(t)\|_P \leq \|z(t) - z^p(t; t_k)\|_P + \|z^p(t; t_k)\|_P$. Following the procedure and results in the proof of Lemma 2, $\|z(t) - z^p(t; t_k)\|_P \leq \Gamma_k \leq \varepsilon/(16\sqrt{N_a})$ for all $t \in [t_k, t_{k+1}]$. Suppose now that $\|z^p(t^2; t_k)\|_P \leq (7\sqrt{N_a} - 1)\varepsilon/(8\sqrt{N_a})$ at some time $t^2 \in [t_k, t_{k+1}]$. Then,

$$\|z(t^2)\|_P \leq \frac{\varepsilon}{16\sqrt{N_a}} + \frac{(14\sqrt{N_a}-1)\varepsilon}{16\sqrt{N_a}} = \frac{7\varepsilon}{8},$$

violating the condition that $\|z(t)\|_P > 7\varepsilon/8$ for all time $t \in [t_k, t_{k+1}]$. Therefore, it can be assumed that $\|z^p(t^2; t_k)\|_P \geq (14\sqrt{N_a} - 1)\varepsilon/(16\sqrt{N_a}) > 13\varepsilon/16$ for all time $t \in [t_k, t_{k+1}]$, using $N_a \geq 10 > 1$. As such,

$$\begin{aligned} -\int_{t_k}^{t_{k+1}} \|z^p(s; t_k)\|_Q^2 \ \mathrm{d}s &\leq -\frac{\delta(13\varepsilon/16)^2 \lambda_{\min}(Q)}{\lambda_{\max}(P)} \\ &\leq -\ln(4N_a)(13\varepsilon/16)^2/4 \end{aligned}$$

Using the Lyapunov function $V_2(\bar{z}(t; t_k), \hat{z}(t; t_k)) = \|\bar{z}(t; t_k)\|_P^2 + \|\hat{z}(t; t_k)\|_P^2$, it is straightforward to show that $\mathrm{d}V_2(\bar{z}(t; t_k), \hat{z}(t; t_k))/\mathrm{d}t \leq -\|\bar{z}(t; t_k)\|_{\widehat{Q}}^2 - \|\hat{z}(t; t_k)\|_{\widehat{Q}}^2$, for all $t \in [t_{k-1}+T, t_k+T]$. Consequently,

$$\int_{t_k+T}^{t_{k+1}+T} \|\bar{z}(s;t_{k+1})\|_Q^2 \mathrm{d}s + \|\bar{z}(t_{k+1}+T;t_{k+1})\|_P^2 - \|z^p(t_k+T;t_k)\|_P^2 \quad \leq \quad \|\bar{z}(t_k+T;t_{k+1})\|_P^2.$$

From the proof of Lemma 4, $\|\bar{z}(t_k+T;t_{k+1})\|_P^2 \leq (3\varepsilon/4)^2$. To bound the remaining terms, observe that

$$\|\bar{z}(t;t_{k+1}) \pm \hat{z}(t;t_{k+1})\|_Q^2 \leq \|\bar{z}(t;t_{k+1}) - \hat{z}(t;t_{k+1})\|_Q^2 + 2\|\bar{z}(t;t_{k+1}) - \hat{z}(t;t_{k+1})\|_Q \|\hat{z}(t;t_{k+1})\|_Q + \|\hat{z}(t;t_{k+1})\|_Q^2.$$

Consequently, it follows that

$$\|\bar{z}(t;t_{k+1})\|_Q^2 - \|\hat{z}(t;t_{k+1})\|_Q^2 \leq \|\bar{z}(t;t_{k+1}) - \hat{z}(t;t_{k+1})\|_Q^2 + 2\|\bar{z}(t;t_{k+1}) - \hat{z}(t;t_{k+1})\|_Q \|\hat{z}(t;t_{k+1})\|_Q.$$

From As. 5, $\|\hat{z}(t;t_{k+1})\|_Q \leq \rho_{\text{max}} \lambda_{\text{max}}^{1/2}(Q)/(\sqrt{N_a}\lambda_{\text{min}}^{1/2}(P))$. From Lemma 2,

$$\|\bar{z}(t;t_{k+1}) - \hat{z}(t;t_{k+1})\|_Q \leq \frac{\varepsilon \lambda_{\text{max}}^{1/2}(Q)}{4\sqrt{N_a}\lambda_{\text{min}}^{1/2}(P)}.$$

Collecting terms, the cost difference inequality becomes

$$J_\Sigma(t_{k+1}) - J_\Sigma(t_k) \leq -\frac{\ln(4N_a)/4(13\varepsilon)^2 - (3\varepsilon)^2}{16} + \frac{\varepsilon(T-\delta)\lambda_{\text{max}}(Q)}{2N_a\lambda_{\text{min}}(P)}\left[\frac{\varepsilon}{8} + \rho_{\text{max}}\right].$$

With $N_a \geq 10$, $\ln(4N_a)/4 > 0.9$, and $0.9(13)^2 > 12^2$. So, it follows that $-\ln(4N_a)/4(13\varepsilon)^2 - (3\varepsilon)^2 < -135\varepsilon^2$. Substitution into the inequality gives

$$J_\Sigma(t_{k+1}) - J_\Sigma(t_k) \leq -C, \quad \text{where,}$$
$$C \triangleq \frac{\varepsilon}{16N_a}\left[135N_a\varepsilon - \frac{(T-\delta)(8\rho_{\text{max}}+\varepsilon)\lambda_{\text{max}}(Q)}{\lambda_{\text{min}}(P)}\right],$$

and (18) ensures that $C > 0$, concluding the proof. ■

Condition (17) requires that, given Q and P, δ must be at least as large as is required by condition (15), since $\lambda_{\text{min}}(Q) \leq \lambda_{\text{min}}(\widehat{Q})$. Condition (18) requires that N_a not be too small, which is consistent with the requirement that $N_a \geq 10$ from (16). Such conditions on N_a are an artifact of the conservative conditions required in the lemmas, e.g., that all agents be within $\Omega_{\varepsilon/(4\sqrt{N_a})}$ to ensure that each agent $i \in \mathcal{V}$ is within $\Omega_i(\varepsilon/(4)$. Outside of the analysis in proving the stated results, there is no clear engineering reason suggesting a lower bound on the number of agents.

To summarize, the system itself dictates the parameters $(\alpha_m, \beta_m, \gamma_m, N_a)$, and the results above require $N_a \geq 10$ (from (16)) to satisfy (18). These system parameters are also involved in conditions (9)–(11), although the primary constraint is the upper bound (10) on β_m, where β_m quantifies a maximal amount of dynamic coupling between subsystems. The design parameters are $(\delta, c_1, T, K, Q, R, W, \varepsilon)$, and P is determined given the linearization and matrices K, Q, R. Feasibility required δ to be not too small nor too big, as quantified by (16) and (17). The control compatibility constraint bound c_1 is also required to be sufficiently small as quantified by (11).

V. CONCLUSIONS

In this paper, a recently developed distributed implementation of receding horizon control is extended to the case of dynamically coupled nonlinear systems. The results, while quite conservative, are only sufficient. A central element to the stability analysis is that the actual and assumed responses of each agent are not too far from one another, as quantified by a control compatibility constraint. Also, the amount of dynamic coupling must sufficiently weak, to permit decoupled terminal controllers and to enable feasibility and stability properties for the algorithm. That the result requires a bound on the amount of coupling is no surprise, while the highly conservative form of the bound itself is an artifact of the sufficient conditions derived here. Less conservative conditions and numerical experiments will be explored in a future work. Relaxations of the theory have recently been employed in the venue of supply chain management [5]. In conclusion, it is noted that the results in this paper are intimately related to those of Michalska and Mayne [10], who demonstrated robustness to model error by placing parametric bounds on (combinations of) the update period and Lipschitz constant. While there is no model error here, bounds are likewise derived to ensure robustness to the bounded discrepancy between what agents do, and what their neighbors believe they will do.

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