Bayesian Modeling, Inference, Prediction and Decision-Making

2b: Integer-Valued Outcomes; Poisson Modeling

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Short course web page:
www.ams.ucsc.edu/∼draper/eBay-Google-2013.html

Integer-Valued Outcomes

Case Study: *Hospital length of stay for birth of premature babies.* As a small part of a study I worked on at the Rand Corporation in the late 1980s, we obtained data on a random sample of $n = 14$ women who came to a hospital in Santa Monica, CA, in 1988 to **give birth to premature babies**.

One (integer-valued) outcome of interest was
$y =$ **length of hospital stay** (LOS).

Here's a preliminary look at the data in the excellent **freeware statistical package** R (see `http://www.r-project.org/` for more details and instructions on how to **download** the package).

```
rosalind 77> R

R : Copyright 2001, The R Development Core Team
Version 1.2.1  (2001-01-15)

R is free software and comes with ABSOLUTELY NO WARRANTY.
You are welcome to redistribute it under certain conditions.
Type 'license()' or 'licence()' for distribution details.

R is a collaborative project with many contributors.
Type 'contributors()' for more information.

Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for a HTML browser interface to help.
Type 'q()' to quit R.

[Previously saved workspace restored]

> y

 [1] 1 2 1 1 1 2 2 4 3 6 2 1 3 0

> sort( y )

 [1] 0 1 1 1 1 1 2 2 2 2 3 3 4 6

> table( y )

0 1 2 3 4 6
1 5 4 2 1 1
```

Poisson Modeling

```
> stem( y, scale = 2 )

  The decimal point is at the |

  0 | 0
  1 | 00000
  2 | 0000
  3 | 00
  4 | 0
  5 |
  6 | 0

> mean( y )

[1] 2.071429

> sqrt( var( y ) )

[1] 1.54244

> q( )

Save workspace image? [y/n/c]: y
rosalind 1777>
```

One possible model for non-negative integer-valued outcomes is the **Poisson distribution**

$$P(Y_i = y_i) = \left\{ \begin{array}{cc} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} & \text{for } y_i = 0, 1, \ldots \\ 0 & \text{otherwise} \end{array} \right\}, \qquad (1)$$

for some $\lambda > 0$.

As usual `Maple` can be used to work out the **mean** and **variance** of this distribution:

```
rosalind 78> maple

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 \  MAPLE  /  reserved. Maple and Maple V are registered trademarks of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
```

Poisson Modeling (continued)

```
> assume( lambda > 0 );

> p := ( y, lambda ) -> lambda^y * exp( - lambda ) / y!;

                                              y
                                         lambda  exp(-lambda)
                    p := (y, lambda) -> ---------------------
                                              y!

> simplify( sum( p( y, lambda ), y = 0 .. infinity ) );

                                         1

> simplify( sum( y * p( y, lambda ), y = 0 .. infinity ) );

                                      lambda~

> simplify( sum( ( y - lambda )^2 * p( y, lambda ),
    y = 0 .. infinity ) );

                                      lambda~
```

Thus if $Y \sim \text{Poisson}(\lambda), E(Y) = V(Y) = \lambda$, which people sometimes express by saying that the **variance-to-mean ratio** (VTMR) for the Poisson is 1.

R can be used to check informally whether the Poisson is a **good fit** to the LOS data:

```
rosalind 77> R

R : Copyright 2001, The R Development Core Team
Version 1.2.1  (2001-01-15)

> dpois( 0:7, mean( y ) )
[1] 0.126005645 0.261011693 0.270333539 0.186658872 0.096662630
[6] 0.040045947 0.013825386 0.004091186

> print( n <- length( y ) )

[1] 14

> table( y ) / n
         0          1          2          3          4          6
0.07142857 0.35714286 0.28571429 0.14285714 0.07142857 0.07142857
```

Poisson Modeling (continued)

```
> cbind( c( dpois( 0:6, mean( y ) ),
    1 - sum( dpois( 0:6, mean( y ) ) ) ),
    apply( outer( y, 0:7, '==' ), 2, sum ) / n )

             [,1]       [,2]
[1,] 0.126005645 0.07142857
[2,] 0.261011693 0.35714286
[3,] 0.270333539 0.28571429
[4,] 0.186658872 0.14285714
[5,] 0.096662630 0.07142857
[6,] 0.040045947 0.00000000
[7,] 0.013825386 0.07142857
[8,] 0.005456286 0.00000000
```

The second column in the above table records the values of the **Poisson probabilities** for $\lambda = 2.07$, the mean of the y_i, and the third column is the **empirical relative frequencies**; informally the fit is reasonably good.

Another **informal check** comes from the fact that the sample mean and variance are 2.07 and $1.542^2 \doteq 2.38$, which are reasonably close.

Exchangeability. As with the AMI mortality case study, before the data arrive I recognize that my uncertainty about the Y_i is exchangeable, and you would expect from a generalization of the binary-outcomes version of de Finetti's Theorem that the structure of a **plausible Bayesian model** for the data might then be

$$\begin{array}{rcll} \theta & \sim & p(\theta) & \text{(prior)} \\ (Y_i|\theta) & \overset{\text{IID}}{\sim} & F(\theta) & \text{(likelihood)}, \end{array} \tag{2}$$

where θ is some parameter (vector) and $F(\theta)$ is some **family of distributions** on the non-negative integers indexed by θ.

Poisson Modeling (continued)

Thus, in view of the preliminary examination of the data above, a **plausible Bayesian model** for these data is

$$\begin{array}{rcll} \lambda & \sim & p(\lambda) & \text{(prior)} \\ (Y_i|\lambda) & \overset{\text{IID}}{\sim} & \text{Poisson}(\lambda) & \text{(likelihood)}, \end{array} \tag{3}$$

where λ is a **positive real number**.

NB (1) This approach to model-building involves a form of **cheating**, because we've **used the data twice**: once to choose the model, and again to draw conclusions conditional on the chosen model.

The result is a failure to **assess** and **propagate model uncertainty** (e.g., Draper 1995).

(2) **Frequentist** modeling often employs this **same kind of cheating** in specifying the likelihood function.

(3) There are two Bayesian ways out of this dilemma: **cross-validation** and **Bayesian nonparametric/semi-parametric** methods (I'll have more to say about both in this course).

To get more practice with Bayesian calculations I'm going to **ignore the model uncertainty problem for now** and pretend that somehow we knew that the Poisson was a good choice.

The likelihood function in model (3) is

$$\begin{aligned} l(\lambda|y) &= c\, p_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n|\lambda) \\ &= c \prod_{i=1}^{n} p_{Y_i}(y_i|\lambda) \\ &= c \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} \\ &= c\, \lambda^s\, e^{-n\lambda}, \end{aligned} \tag{4}$$

The Conjugate Prior

where $y = (y_1, \ldots, y_n)$ and $s = \sum_{i=1}^n y_i$; here $\left(\prod_{i=1}^n y_i!\right)^{-1}$ can be **ignored** because it doesn't involve λ.

Thus (as was true in the Bernoulli model) $s = \sum_{i=1}^n y_i$ is **sufficient** for λ in the Poisson model, and we can write $l(\lambda|s)$ instead of $l(\lambda|y)$ if we want.

If a **conjugate** prior $p(\lambda)$ for λ exists it must be such that the product $p(\lambda)\, l(\lambda|s)$ has the same mathematical form as $p(\lambda)$.

Examination of (4) reveals that the same trick works here as with Bernoulli data, namely taking the **prior to be of the same form as the likelihood**:

$$p(\lambda) = c\, \lambda^{\alpha-1} e^{-\beta\lambda} \tag{5}$$

for some $\alpha > 0, \beta > 0$ — this is the **Gamma** distribution $\lambda \sim \Gamma(\alpha, \beta)$ for $\lambda > 0$ (see Gelman et al. Appendix A).

As usual `Maple` can work out the **normalizing constant**:

```
rosalind 80> maple

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 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.

> assume( lambda > 0, alpha > 0, beta > 0 );

> p1 := ( lambda, alpha, beta ) -> lambda^( alpha - 1 ) *
    exp( - beta * lambda );

                                              (alpha - 1)
      p1 := (lambda, alpha, beta) -> lambda              exp(-beta lambda)

> simplify( integrate( p1( lambda, alpha, beta ),
    lambda = 0 .. infinity ) );

                                (-alpha~)
                          beta~          GAMMA(alpha~)
```

The Gamma Distribution

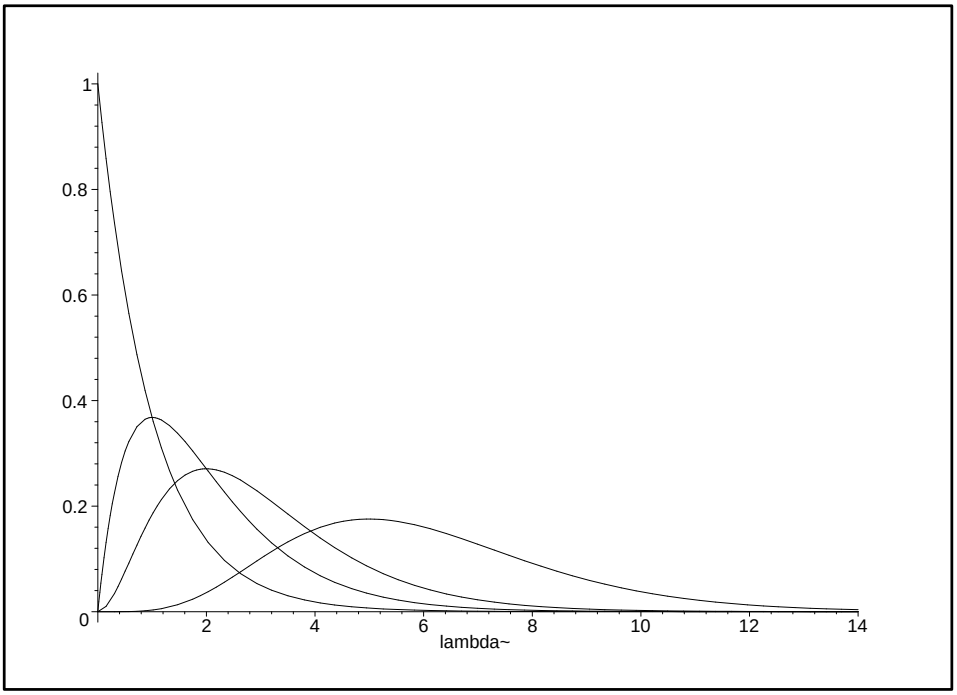
Thus $c^{-1} = \beta^{-\alpha}\,\Gamma(\alpha)$ and the
proper definition of the Gamma distribution is

$$\text{If } \lambda \sim \Gamma(\alpha, \beta) \text{ then } p(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\,\lambda^{\alpha-1}\,e^{-\beta\,\lambda} \qquad (6)$$

for $\alpha > 0, \beta > 0$.

As usual `Maple` can be used to explore the behavior of this family of distributions **as a function of its inputs** α and β:

```
> p := ( lambda, alpha, beta ) -> beta^alpha * lambda^( alpha - 1 ) *
    exp( - beta * lambda ) / GAMMA( alpha );

                                    alpha       (alpha - 1)
                                beta      lambda             exp(-beta lambda)
p := (lambda, alpha, beta) -> ----------------------------------------------
                                             GAMMA(alpha)

> plotsetup( x11 );

> plot( { p( lambda, 1, 1 ), p( lambda, 2, 1 ), p( lambda, 3, 1 ),
    p( lambda, 6, 1 ) }, lambda = 0 .. 14, color = black );
```



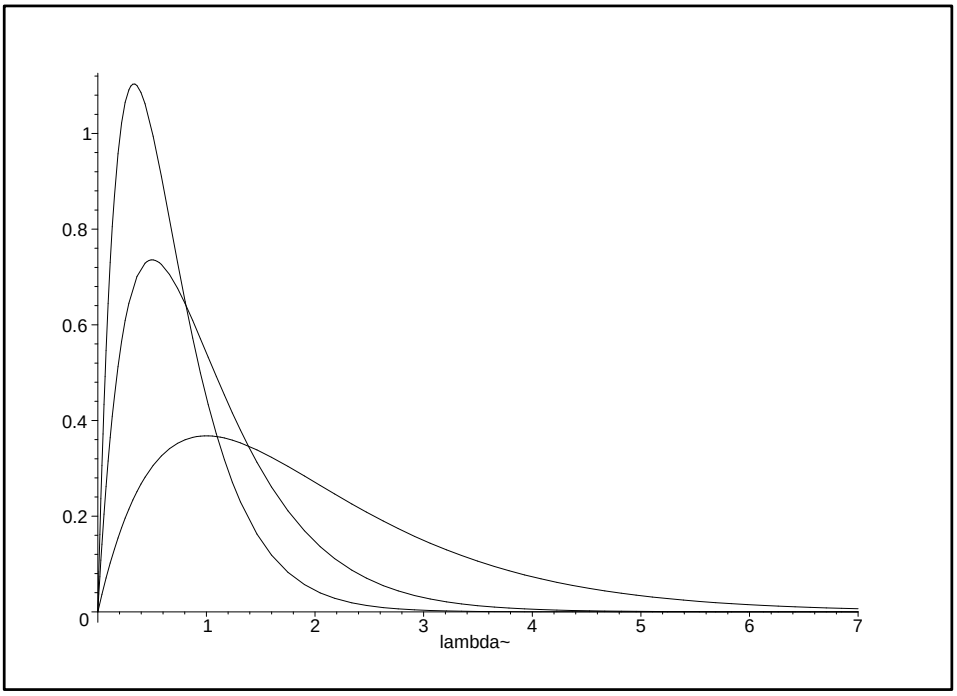
α evidently controls the **shape** of the Gamma family.

Gamma Distribution (continued)

When $\alpha = 1$ the Gamma distributions have a special form that you'll probably recognize — they're the **exponential** distributions $\mathcal{E}(\beta)$: for $\beta > 0$

$$\text{If } \lambda \sim \mathcal{E}(\beta) \text{ then } p(\lambda) = \left\{ \begin{array}{cc} \beta\, e^{-\beta\,\lambda} & \text{for } \lambda > 0 \\ 0 & \text{otherwise} \end{array} \right\}. \qquad (7)$$

```
> plot( { p( lambda, 2, 1 ), p( lambda, 2, 2 ), p( lambda, 2, 3 ) },
    lambda = 0 .. 7, color = black );
```



In the Gamma family the parameter β controls the **spread** or **scale** of the distribution.

Definition Given a random quantity y whose density $p(y|\sigma)$ depends on a parameter $\sigma > 0$, if it's possible to express $p(y|\sigma)$ in the form $\frac{1}{\sigma}\, f(\frac{y}{\sigma})$, where $f(\cdot)$ is a function that does not depend on y or σ, then σ is called a **scale** parameter for the parametric family p.

Gamma Distribution (continued)

Letting $f(t) = e^{-t}$ and taking $\sigma = \frac{1}{\beta}$, you can see that the Gamma family can be expressed in this way, so $\frac{1}{\beta}$ is a **scale parameter** for the Gamma distribution.

As usual `Maple` can also work out the **mean** and **variance** of this family:

```
> simplify( integrate( p( lambda, alpha, beta ),
    lambda = 0 .. infinity ) );

                                  1

> simplify( integrate( lambda * p( lambda, alpha, beta ),
    lambda = 0 .. infinity ) );

                                alpha~
                                ------
                                beta~


> simplify( integrate( ( lambda - alpha / beta )^2 *
    p( lambda, alpha, beta ), lambda = 0 .. infinity ) );

                                alpha~
                                ------
                                     2
                                beta~
```

Thus if $\lambda \sim \Gamma(\alpha, \beta)$ then $E(\lambda) = \frac{\alpha}{\beta}$ and $V(\lambda) = \frac{\alpha}{\beta^2}$.

Conjugate updating is now **straightforward**: with $y = (y_1, \ldots, y_n)$ and $s = \sum_{i=1}^{n} y_i$, by Bayes' Theorem

$$\begin{aligned} p(\lambda|y) &= c\, p(\lambda)\, l(\lambda|y) \\ &= c \left(c\, \lambda^{\alpha-1}\, e^{-\beta\lambda}\right) \left(c\, \lambda^{s}\, e^{-n\lambda}\right) \\ &= c\, \lambda^{(\alpha+s)-1}\, e^{-(\beta+n)\lambda}, \end{aligned} \tag{8}$$

and the **resulting distribution** is just $\Gamma(\alpha + s, \beta + n)$.

Conjugate Poisson Analysis

This can be **summarized** as follows:

$$\left\{ \begin{array}{c} (\lambda|\alpha,\beta) \sim \Gamma(\alpha,\beta) \\ (Y_i|\lambda) \overset{\text{IID}}{\sim} \text{Poisson}(\lambda), \\ i = 1,\ldots,n \end{array} \right\} \rightarrow (\lambda|s) \sim \Gamma(\alpha^*,\beta^*), \qquad (9)$$

where $(\alpha^*,\beta^*) = (\alpha+s,\beta+n)$ and $s = \sum_{i=1}^n y_i$ is a **sufficient statistic** for λ in this model.

The posterior mean of λ here is evidently $\frac{\alpha^*}{\beta^*} = \frac{\alpha+s}{\beta+n}$, and the prior and data means are $\frac{\alpha}{\beta}$ and $\bar{y} = \frac{s}{n}$, so (as was the case in the Bernoulli model) the posterior mean can be written as a **weighted average** of the prior and data means:

$$\frac{\alpha+s}{\beta+n} = \left(\frac{\beta}{\beta+n}\right)\left(\frac{\alpha}{\beta}\right) + \left(\frac{n}{\beta+n}\right)\left(\frac{s}{n}\right). \qquad (10)$$

Thus the **prior sample size** n_0 in this model is just β (which makes sense given that $\frac{1}{\beta}$ is the scale parameter for the Gamma distribution), and the prior acts like a **dataset** consisting of β observations with mean $\frac{\alpha}{\beta}$.

LOS data analysis. Suppose that, before the current data set is scheduled to arrive, I know **little** about the mean length of hospital stay of women giving birth to premature babies.

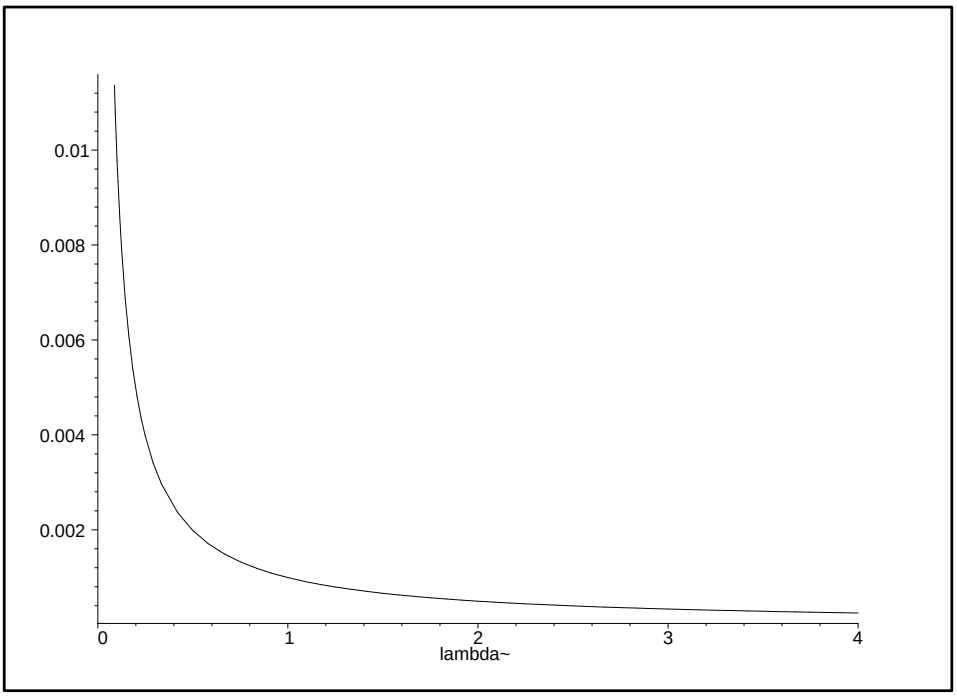
Then for my prior on λ I'd like to specify a member of the $\Gamma(\alpha,\beta)$ family that's relatively **flat in the region in which the likelihood function is appreciable**.

The $\Gamma(\epsilon, \epsilon)$ Prior

A **convenient** and **fairly all-purpose default choice** of this type is $\Gamma(\epsilon, \epsilon)$ for some small ϵ like 0.001.

When used as a prior this distribution has **prior sample size** ϵ; it also has mean 1, but that usually doesn't matter when ϵ is **tiny**.

```
> plot( p( lambda, 0.001, 0.001 ), lambda = 0 .. 4, color = black );
```



With the LOS data $s = 29$ and $n = 14$, so the **likelihood** for λ is like a $\Gamma(30, 14)$ density, which has mean $\frac{30}{14} \doteq 2.14$ and

$$\text{SD } \sqrt{\frac{30}{14^2}} \doteq 0.39.$$

Thus by the **Empirical Rule** the likelihood is appreciable in the range $(\text{mean} \pm 3\,\text{SD}) \doteq (2.14 \pm 1.17) \doteq (1.0, 3.3)$, and you can see from the plot above that the prior is indeed **relatively flat** in this region.

From the **Bayesian updating** in (9), with a $\Gamma(0.001, 0.001)$ prior the **posterior** is $\Gamma(29.001, 14.001)$.

LOS Data Analysis

It's useful, in summarizing the **updating** from prior through likelihood to posterior, to make a table that records measures of **center** and **spread** at each point along the way.

For example, the $\Gamma(0.001, 0.001)$ **prior**, when regarded (as usual) as a **density** for λ, has mean 1.000 and SD $\sqrt{1000} \doteq 31.6$ (i.e., informally, as far as we're concerned, before the data arrive λ could be **anywhere between 0 and (say) 100**).

And the $\Gamma(29.001, 14.001)$ **posterior** has mean $\frac{29.001}{14.001} \doteq 2.071$ and SD $\sqrt{\frac{29.001}{14.001^2}} \doteq 0.385$, so after the data have arrived we know **quite a bit more than before**.

There are two main ways to summarize the **likelihood** — Fisher's approach based on **maximizing** it, and the Bayesian approach based on regarding it as a density and **integrating** it — and it's instructive to compute them both and **compare**.

The **likelihood-integrating** approach (which is actually equivalent to another of Fisher's ideas: **fiducial** inference) treats the $\Gamma(30, 14)$ likelihood as a density for λ, with mean $\frac{30}{14} \doteq 2.143$ and SD $\sqrt{\frac{30}{14^2}} \doteq 0.391$.

As for the **likelihood-maximizing** approach, from (4) the log likelihood function is

$$ll(\lambda|y) = ll(\lambda|s) = \log\left(c\,\lambda^s e^{-n\lambda}\right) = c + s\log\lambda - n\lambda, \qquad (11)$$

and this is **maximized** as usual (check that it's the max) by setting the **derivative** equal to 0 and solving:

$$\frac{\partial}{\partial\lambda} ll(\lambda|s) = \frac{s}{\lambda} - n = 0 \quad \text{iff} \quad \lambda = \hat{\lambda}_{\text{MLE}} = \frac{s}{n} = \bar{y}. \qquad (12)$$

LOS Analysis (continued)

Since the MLE $\hat{\lambda}_{\text{MLE}}$ turns out to be our old friend the **sample mean** $\bar{y}$, you might be tempted to conclude immediately that $\widehat{SE}\big(\hat{\lambda}_{\text{MLE}}\big) = \frac{\hat{\sigma}}{\sqrt{n}}$, where $\hat{\sigma} = 1.54$ is the sample SD, and indeed it's true in repeated sampling that $V\big(\bar{Y}\big) = \frac{V(Y_1)}{n}$; but the **Poisson distribution** has variance $V(Y_1) = \lambda$, so that $\sqrt{V\big(\bar{Y}\big)} = \frac{\sqrt{\lambda}}{\sqrt{n}}$, and there's no guarantee in the Poisson model that the best way to estimate $\sqrt{\lambda}$ in this standard error calculation is with the sample SD $\hat{\sigma}$ (in fact we have a **strong hint** from the above MLE calculation that the sample variance is **irrelevant** to the estimation of λ in the Poisson model).

The correct (large-sample) likelihood-based **standard error** for $\hat{\lambda}_{\text{MLE}}$, using the **Fisher information** logic we examined earlier, is obtained from the following calculation:

$$\begin{aligned} \frac{\partial^2}{\partial \lambda^2} \log l(\lambda|y) &= -\frac{s}{\lambda^2}, \quad \text{so} \qquad (13) \\ \hat{I}\big(\hat{\lambda}_{\text{MLE}}\big) &= \left[-\frac{\partial^2}{\partial \lambda^2} \log l(\lambda|y)\right]_{\lambda=\hat{\lambda}_{\text{MLE}}} \\ &= \left(\frac{s}{\lambda^2}\right)_{\lambda=\bar{y}} = \frac{s}{\bar{y}^2} = \frac{n}{\bar{y}}, \quad \text{and} \\ \hat{V}\big(\hat{\lambda}_{\text{MLE}}\big) &= \hat{I}^{-1}\big(\hat{\lambda}_{\text{MLE}}\big) = \frac{\bar{y}}{n} = \frac{\hat{\lambda}_{\text{MLE}}}{n}. \end{aligned}$$

So in this case study Fisher's **likelihood-maximizing** approach would **estimate** λ by $\hat{\lambda}_{\text{MLE}} = \bar{y} = \frac{29}{14} \doteq 2.071$, with a **give-or-take** of $\widehat{SE}\big(\hat{\lambda}_{\text{MLE}}\big) = \frac{\sqrt{\hat{\lambda}_{\text{MLE}}}}{\sqrt{n}} = \frac{1.44}{\sqrt{14}} \doteq 0.385$.

LOS Analysis (continued)

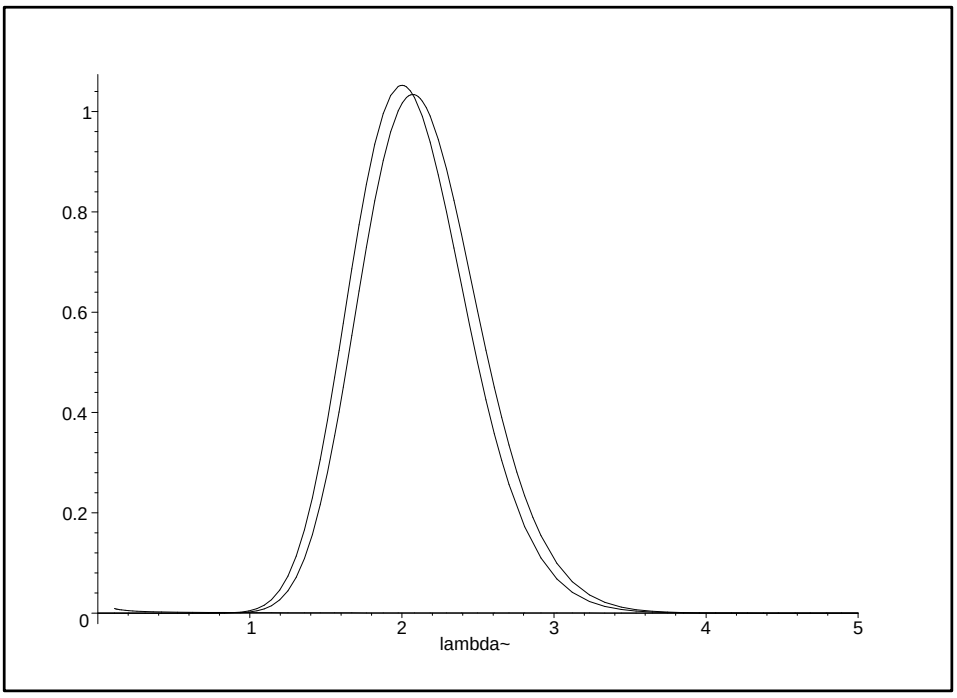
All of this may be **summarized** in the following table:

		Likelihood		
	Prior	Maximizing	Integrating	Posterior
Mean/Estimate	1.00	2.071	2.143	2.071
SD/SE	31.6	0.385	0.391	0.385

The **discrepancies** between the likelihood-maximizing and likelihood-integrating columns in this table would be smaller with a larger sample size and would **tend to 0** as $n \to \infty$.

The **prior-likelihood-posterior plot** comes out like this:

```
> plot( { p( lambda, 0.001, 0.001 ), p( lambda, 30, 14 ),
    p( lambda, 29.001, 14.001 ) }, lambda = 0 .. 5, color = black );
```



LOS Analysis (continued)

For **interval estimation** in the maximum-likelihood approach the best we could do, using the technology I've described to you so far, would be to appeal to the **CLT** (even though n is only 14) and use $\hat{\lambda}_{\mathrm{MLE}} \pm 1.96\,\widehat{SE}(\hat{\lambda}_{\mathrm{MLE}}) \doteq 2.071 \pm (1.96)(0.385) \doteq (1.316, 2.826)$ as an **approximate 95% confidence interval** for λ.

You can see from the previous plot that the likelihood function is **skewed**, so a more careful method (e.g., the **bootstrap**; Efron 1979) would be needed to create a better interval estimate from the likelihood point of view.

Some trial and error with `Maple` can be used to find the lower and upper limits of the **central 95% posterior interval** for λ:

```
> evalf( Int( p( lambda, 29.001, 14.001 ), lambda = 0 .. 1.316 ) );

                                .01365067305

> evalf( Int( p( lambda, 29.001, 14.001 ), lambda = 0 .. 1.4 ) );

                                .02764660367

> evalf( Int( p( lambda, 29.001, 14.001 ), lambda = 0 .. 1.387 ) );

                                .02495470339

> evalf( Int( p( lambda, 29.001, 14.001 ), lambda = 2.826 .. infinity ) );

                                .03403487851

> evalf( Int( p( lambda, 29.001, 14.001 ), lambda = 2.890 .. 5 ) );

                                .02505306648

> evalf( Int( p( lambda, 29.001, 14.001 ),
    lambda = 2.890 .. infinity ) );

                                .02505307631
```

LOS Analysis (continued)

Thus a **95% (central) posterior interval** for λ, given a diffuse prior, runs from **1.387** to **2.890**, and is (correctly) **asymmetric** around the posterior mean of 2.071.

R can be used to work out the **limits of this interval** even more readily:

```
> help( qgamma )
GammaDist                package:base                R Documentation

The Gamma Distribution

Description:

     Density, distribution function, quantile function and random
     generation for the Gamma distribution with parameters shape and
     scale.

Usage:

     dgamma(x, shape, rate = 1, scale = 1/rate, log = FALSE)
     pgamma(q, shape, rate = 1, scale = 1/rate, lower.tail = TRUE,
       log.p = FALSE)
     qgamma(p, shape, rate = 1, scale = 1/rate, lower.tail = TRUE,
       log.p = FALSE)
     rgamma(n, shape, rate = 1, scale = 1/rate)

Arguments:

    x, q: vector of quantiles.

       p: vector of probabilities.

       n: number of observations. If length(n) > 1, the length is
       taken to be the number required.

shape, scale: shape and scale parameters. Must be positive, scale
strictly.

log, log.p: logical; if TRUE, probabilities p are given as log(p).

lower.tail: logical; if TRUE (default), probabilities are P[X <= x],
          otherwise, P[X > x].
```

LOS Analysis (continued)

```
Details:

     The Gamma distribution with parameters shape = a and scale = s
     has density

                f(x)= 1/(s^a Gamma(a)) x^(a-1) e^-(x/s)

     for x > 0, a > 0 and s > 0. The mean and variance are E(X) = a*s
     and Var(X) = a*s^2.
```

Warning There are **two common parameterizations** of the **Gamma distribution**: one is as in **equation (6)** above, in which $\frac{1}{\beta}$ is the **scale parameter**, and one as in `R`, in which the **scale parameter** `s` is **equivalent** to $\frac{1}{\beta}$ in the **parameterization** I'm using.

The **easiest way** to see **which parameterization** somebody is **using** is to **ask them** to **write down** the **Gamma density function**: if it **looks** like $p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$, then $\frac{1}{\beta}$ is the **scale parameter**; if it **looks** like

```
f(x)= 1/(s^a Gamma(a)) x^(a-1) e^-(x/s)
```

then `s` is the **scale parameter**.

The other **dead giveaway** is to **ask** what the **mean** and **variance** are: if the **person** is **using (6)**, the **answers** are $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta^2}$ (**respectively**), whereas if the **person** is **using the** `R` **representation** then the **mean** and **variance** are $a * s$ and $a * s^2$ (**respectively**).

```
> qgamma( 0.025, 29.001, scale = 1 / 14.001 )

[1] 1.387228

> qgamma( 0.975, 29.001, scale = 1 / 14.001 )

[1] 2.890435
```

LOS Analysis (continued)

Maple or R can also be used to obtain the **probability content**, according to the posterior distribution, of the approximate 95% (large-sample) likelihood-based interval:

```
> evalf( Int( p( lambda, 29.001, 14.001 ), lambda = 1.316 .. 2.826 ) );

                                .9523144484
```

So the **maximization** approach has led to **decent approximations** here (later I'll give examples where maximum likelihood doesn't do so well in small samples).

Predictive distributions in this model can be computed by Maple **in the usual way**: for instance, to compute $p(y_{n+1}|y)$ for $y = (y_1, \ldots, y_n)$ we want to evaluate

$$\begin{aligned} p(y_{n+1}|y) &= \int_0^\infty p(y_{n+1}, \lambda|y)\, d\lambda \\ &= \int_0^\infty p(y_{n+1}|\lambda, y)\, p(\lambda|y)\, d\lambda \qquad (14) \\ &= \int_0^\infty p(y_{n+1}|\lambda)\, p(\lambda|y)\, d\lambda \\ &= \int_0^\infty \frac{\lambda^{y_{n+1}} e^{-\lambda}}{y_{n+1}!} \frac{(\beta^*)^{\alpha^*}}{\Gamma(\alpha^*)} \lambda^{\alpha^*-1} e^{-\beta^*\lambda}\, d\lambda, \\ &= \frac{(\beta^*)^{\alpha^*}}{\Gamma(\alpha^*)\, y_{n+1}!} \int_0^\infty \lambda^{(\alpha^*+y_{n+1})-1} e^{-(\beta^*+1)\lambda}\, d\lambda, \end{aligned}$$

where $\alpha^* = \alpha + s$ and $\beta^* = \beta + n$; in these expressions y_{n+1} is a **non-negative integer**.

```
> assume( astar > 0, bstar > 0, yf > 0 );

> simplify( bstar^astar * int( lambda^( astar + yf - 1 ) *
    exp( - ( bstar + 1 ) * lambda ), lambda = 0 .. infinity ) /
    ( GAMMA( astar ) * yf! ) );

                astar~                (-astar~ - yf~)
          bstar~       (bstar~ + 1)                  GAMMA(astar~ + yf~)
          ------------------------------------------------------------
                      GAMMA(astar~) GAMMA(yf~ + 1)
```

Predictive Distributions

A bit of **rearranging** then gives that for $y_{n+1} = 0, 1, \ldots,$

$$p(y_{n+1}|y) = \frac{\Gamma(\alpha^* + y_{n+1})}{\Gamma(\alpha^*)\,\Gamma(y_{n+1}+1)} \left(\frac{\beta^*}{\beta^*+1}\right)^{\alpha^*} \left(\frac{1}{\beta^*+1}\right)^{y_{n+1}}. \quad (15)$$

This is called the **Poisson-Gamma** distribution, because (14) is asking us to take a **mixture** (weighted average) of Poisson distributions, using probabilities from a Gamma distribution as the mixing weights.

(15) is a generalization of the **negative binomial** distribution (e.g., Johnson and Kotz 1994), which you've probably encountered in your earlier study of probability and/or statistics.

`Maple` can try to get simple expressions for the **mean** and **variance** of this distribution:

```
> pg := ( y, alpha, beta ) -> GAMMA( alpha + y ) *
    ( beta / ( beta + 1 ) )^alpha * ( 1 / ( beta + 1 ) )^y /
    ( GAMMA( alpha ) * GAMMA( y + 1 ) );

                                                 /  beta  \alpha /   1    \y
                                    GAMMA(alpha + y) |-------|      |-------|
                                                     \beta + 1/     \beta + 1/
  pg := (y, alpha, beta) -> ----------------------------------------------
                                        GAMMA(alpha) GAMMA(y + 1)

> simplify( sum( pg( y, alpha, beta ), y = 0 .. infinity ) );

                                          1

> simplify( sum( y * pg( y, alpha, beta ), y = 0 .. infinity ) );

                                       alpha
                                       -----
                                       beta
```

So the **mean** of the distribution in (15) is $E(y_{n+1}|y) = \frac{\alpha^*}{\beta^*}$.

Inference and Prediction

```
> simplify( sum( ( y - alpha / beta )^2 * pg( y, alpha, beta ),
    y = 0 .. infinity ) );

     2 /  beta  \alpha                          alpha - beta    alpha - beta
alpha  |--------|      hypergeom([alpha, - ------------, - ------------],
       \beta + 1/                               beta            beta

       alpha    alpha      1         /     2
    [- -----, - -----], --------)  /  beta
       beta     beta    beta + 1  /
```

Maple has failed to realize that this expression may be **considerably simplified**: Bernardo and Smith (1994) note that the variance of the distribution in (15) is just

$$V(y_{n+1}|y) = \frac{\alpha^*}{\beta^*}\left(1 + \frac{1}{\beta^*}\right). \tag{16}$$

This provides an interesting **contrast** between **inference** and **prediction**: we've already seen in this model that the posterior mean and variance of λ are $\frac{\alpha^*}{\beta^*} = \frac{\alpha+s}{\beta+n}$ and $\frac{\alpha^*}{(\beta^*)^2} = \frac{\alpha+s}{(\beta+n)^2}$, respectively.

	Posterior	
Quantity	Mean	Variance
λ	$\frac{\alpha+s}{\beta+n}$	$\frac{\alpha+s}{(\beta+n)^2} = \frac{\alpha+s}{\beta+n}\left(0 + \frac{1}{\beta+n}\right)$
y_{n+1}	$\frac{\alpha+s}{\beta+n}$	$\frac{\alpha+s}{\beta+n}\left(1 + \frac{1}{\beta+n}\right)$

Thus λ (the **inferential** objective) and y_{n+1} (the **predictive** objective) have the same posterior mean, but the posterior variance of y_{n+1} is **much larger**, as can be seen by the following argument.

(1) Denoting by μ the mean of the **population** from which the Y_i are thought of as (like) a random sample, when n is large α and β will be **small** in relation to s and n, respectively, and the ratio $\bar{y} = \frac{s}{n}$ should **more and more closely approach** μ — thus for large n,

$$E(\lambda|y) = E(y_{n+1}|y) \doteq \mu. \tag{17}$$

Inference and Prediction

(2) For the Poisson distribution the (population) mean μ and variance σ^2 are **equal**, meaning that for large n the ratio $\frac{\alpha+s}{\beta+n}$ will be close both to μ and to σ^2.

Thus for large n,

$$V(\lambda|y) \doteq \frac{\sigma^2}{n} \quad \text{but} \quad V(y_{n+1}|y) \doteq \sigma^2. \tag{18}$$

An informal way to restate (18) is to say that accurate **prediction** of new data is an **order of magnitude harder** (in powers of n) than accurate **inference** about population parameters.

Bayesian model-checking with predictive distributions.

One way to **check** a model like (1) is as follows.

```
for ( i in 1:n ) {
```

Temporarily **set aside** observation y_i, obtaining a new dataset $y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ with $(n-1)$ observations.

Use the current Bayesian model applied to y_{-i} to **predict** y_i, and summarize the extent to which the actual value of y_i is **surprising** in view of this predictive distribution.

```
}
```

One possible measure of surprise is **predictive z–scores**:

$$z_i = \frac{y_i - E[y_i|y_{-i}]}{\sqrt{V[y_i|y_{-i}]}}. \tag{19}$$

Compare the surprise measure with its **expected behavior** if the model had been **"correct"** (e.g., $z = (z_1, \ldots, z_n)$ should have mean 0 and SD 1).

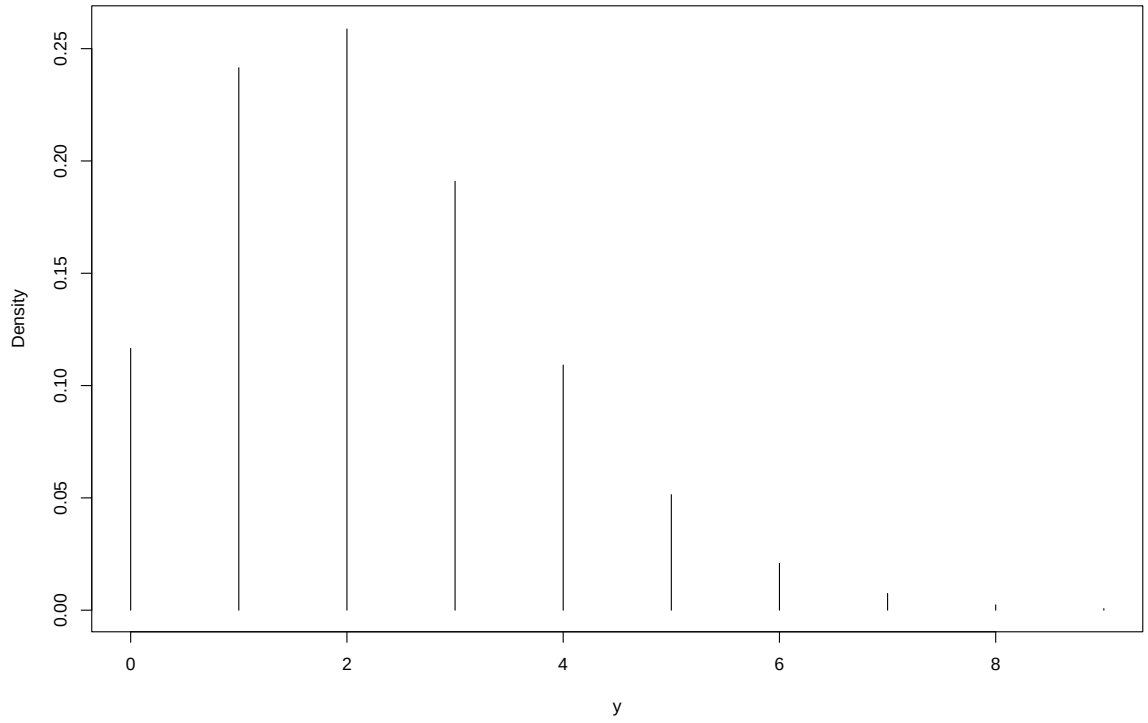
Predictive Model-Checking

Example: the LOS data. Here's some R code to carry out this program on the **LOS data**.

```
rosalind 25> R

R : Copyright 2001, The R Development Core Team
Version 1.2.1  (2001-01-15)

> poisson.gamma <- function( y, alpha, beta ) {

    log.density <- lgamma( alpha + y ) + alpha *
      log( beta / ( beta + 1 ) ) + y * log( 1 / ( beta + 1 ) ) -
      lgamma( alpha ) - lgamma( y + 1 )

    return( exp( log.density ) )

  }

> print( y <- sort( y ) )

 [1] 0 1 1 1 1 1 2 2 2 2 3 3 4 6

> print( y.current <- y[ -1 ] )

 [1] 1 1 1 1 1 2 2 2 2 3 3 4 6

> print( n.current <- length( y.current ) )

[1] 13

> alpha <- beta <- 0.001

> print( s.current <- sum( y.current ) )

[1] 29

> print( alpha.star <- alpha + s.current )

[1] 29.001

> print( beta.star <- beta + n.current )

[1] 13.001
```

Predictive Model-Checking

```
> print( pg.current <- poisson.gamma( 0:9, alpha.star, beta.star ) )

 [1] 0.1165953406 0.2415099974 0.2587508547 0.1909752933 0.1091243547
 [6] 0.0514422231 0.0208209774 0.0074357447 0.0023899565 0.0007017815

> postscript( "pg1.ps" )

> plot( 0:9, pg.current, type = 'n', xlab = 'y', ylab = 'Density' )

> for ( i in 0:9 ) {

    segments( i, 0, i, pg.current[ i + 1 ] )

  }

> dev.off( )

null device
          1
```



The omitted observed value of **0** is **not too unusual** in this predictive distribution.

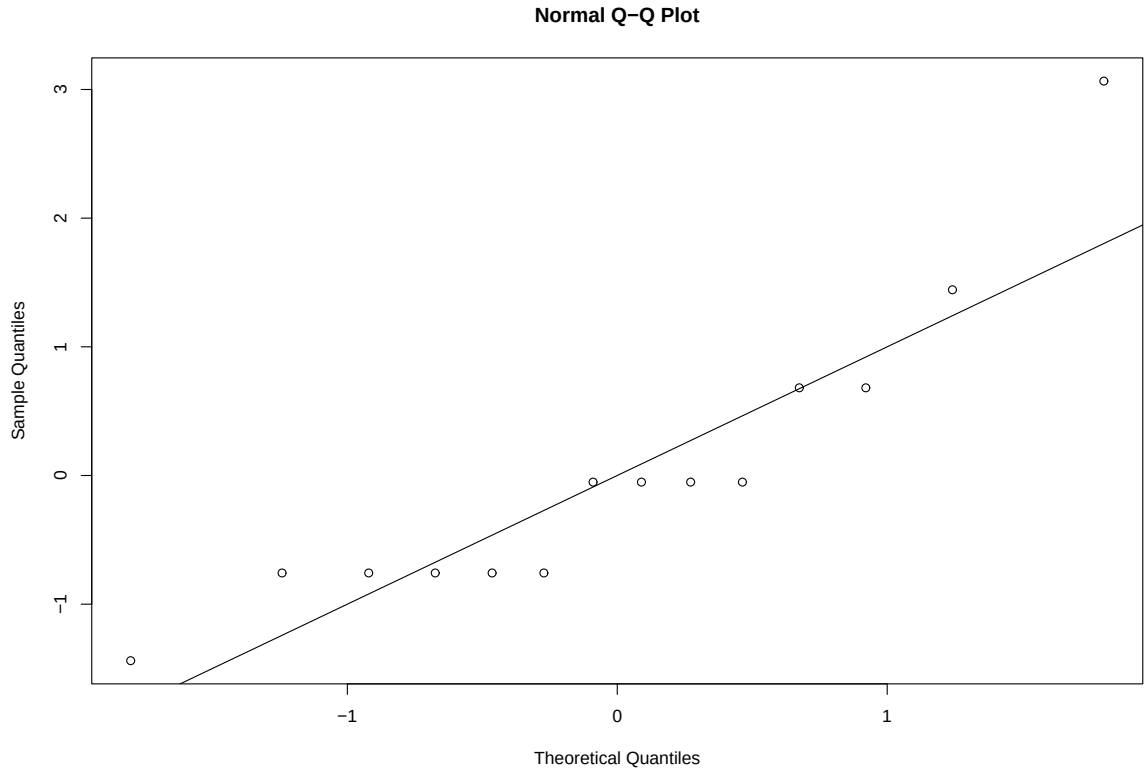
Predictive Model-Checking

The following R code **loops** through the whole dataset to get the **predictive z–scores**.

```
alpha <- beta <- 0.001

z <- rep( 0, n )

for ( i in 1:n ) {

  y.current <- y[ -i ]

  n.current <- length( y.current )

  s.current <- sum( y.current )

  alpha.star <- alpha + s.current

  beta.star <- beta + n.current

  predictive.mean.current <- alpha.star / beta.star

  predictive.SD.current <- sqrt( ( alpha.star / beta.star ) *
    ( 1 + 1 / beta.star ) )

  z[ i ] <- ( y[ i ] - predictive.mean.current ) /
    predictive.SD.current

}

> z

 [1] -1.43921925 -0.75757382 -0.75757382 -0.75757382 -0.75757382
 [6] -0.75757382 -0.05138023 -0.05138023 -0.05138023 -0.05138023
[11]  0.68145253  0.68145253  1.44329065  3.06513271

> mean( z )

[1] 0.03133708

> sqrt( var( z ) )

[1] 1.155077
```

Predictive Model-Checking

```
> postscript( "pg2.ps" )

> qqnorm( z )

> abline( 0, 1 )
```



The 14 predictive z–scores have mean **0.03** (about right) and SD **1.16** (close enough to 1 when sampling variability is considered?), and the **normal qqplot** above shows that the only really surprising observation in the data, as far as the Poisson model was concerned, is the value of **6**, which has a z–score of **3.07**.

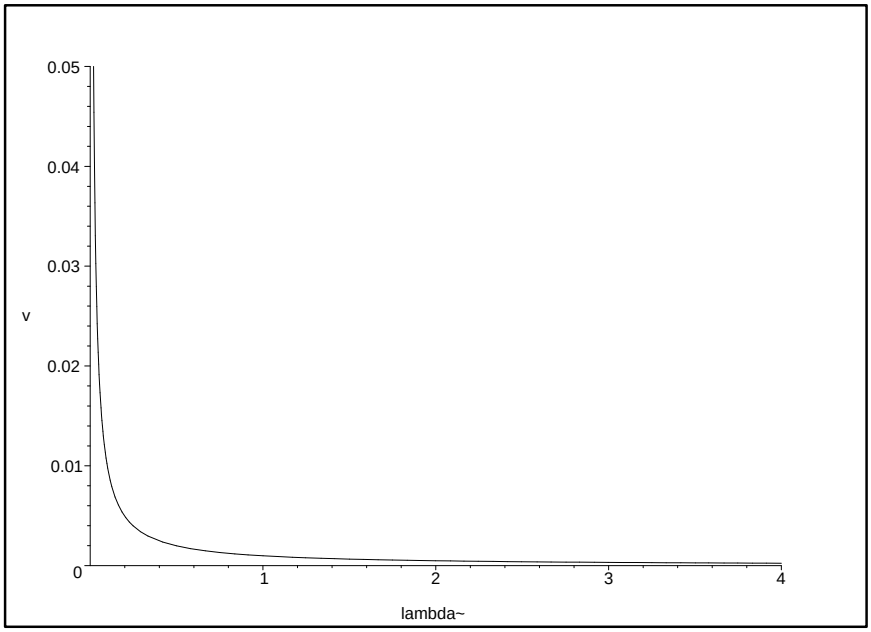
NB The figure above is only a **crude approximation** to the right qqplot, which would have to be created by **simulation**; even so it's enough to **suggest** how the model might be **improved**.

I would conclude **informally** (a) that the Poisson is a **decent** model for these data, but (b) if you wanted to expand the model in a direction suggested by this diagnostic you should look for a model with **extra-Poisson variation**: the sample VTMR in this dataset was about **1.15**.

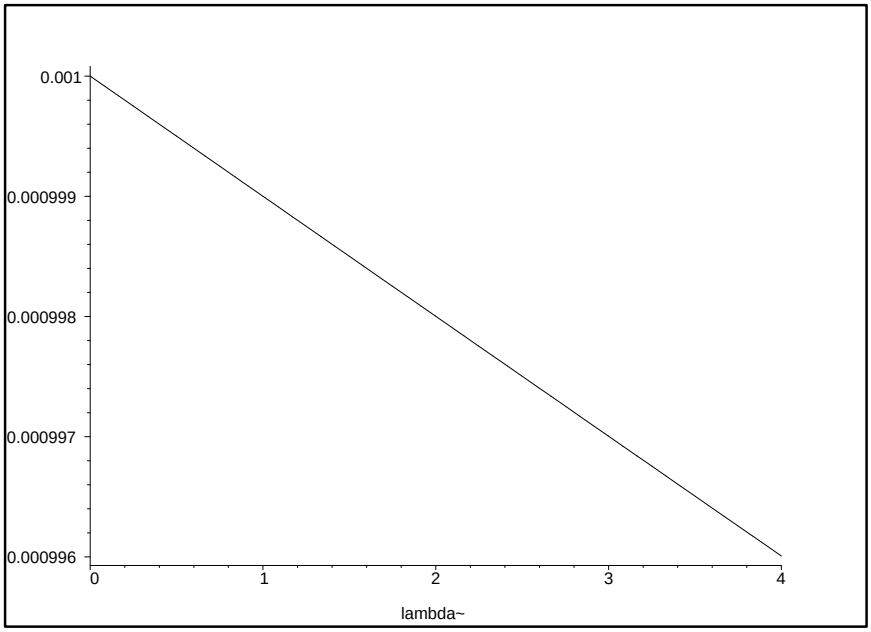
Diffuse Priors in the LOS Case Study

In specifying a **diffuse** prior for λ in the LOS case study, several **alternatives** to $\Gamma(\epsilon, \epsilon)$ might occur to you, including $\Gamma(1, \epsilon), \Gamma(\alpha, \beta)$ for some large α (like 20, to get a roughly **normal** prior) and small β (like 1, to have a **small prior sample size**), and $U(0, C)$ for some cutoff C (like 4) chosen to avoid **truncation** of the likelihood function, where $U(a, b)$ denotes the **uniform** distribution on (a, b).

```
> plot( p( lambda, 0.001, 0.001 ), lambda = 0 .. 4, v = 0 .. 0.05,
    color = black );
```



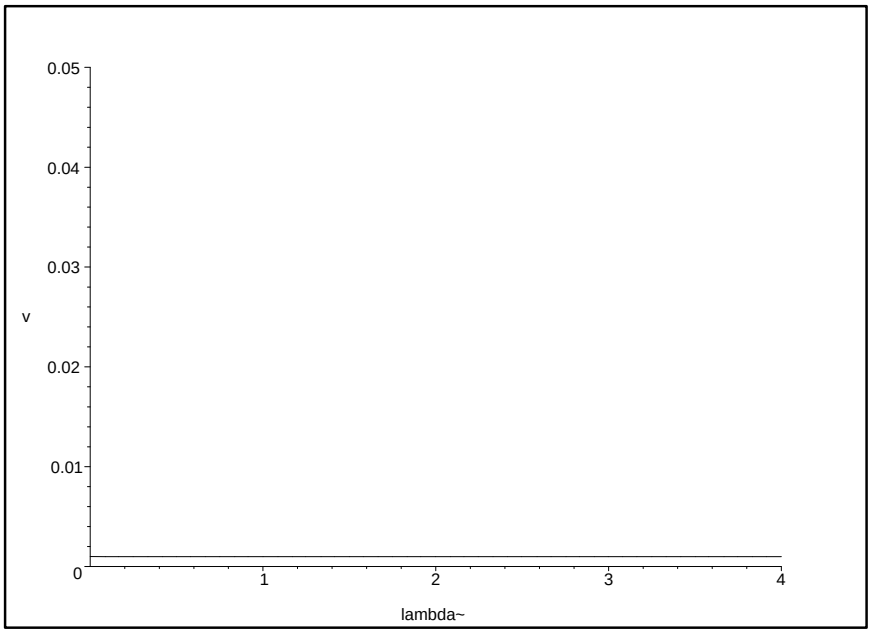

```
> plot( p( lambda, 1.0, 0.001 ), lambda = 0 .. 4, color = black );
```



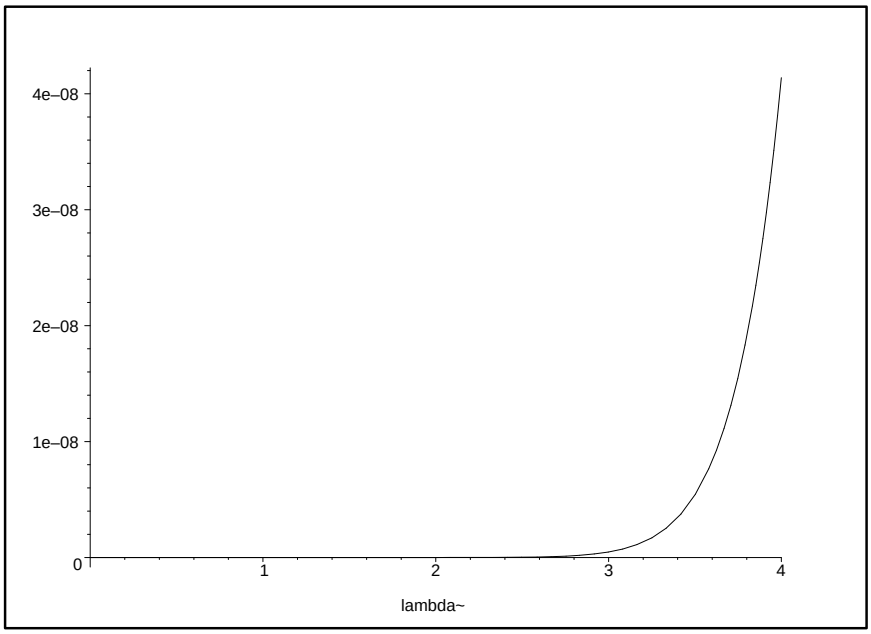
Diffuse Priors (continued)

$\Gamma(1, \epsilon)$ doesn't look promising initially as a **flat** prior, but that's a consequence of `Maple`'s default choice of **vertical axis**.

```
> plot( p( lambda, 1.0, 0.001 ), lambda = 0 .. 4, v = 0 .. 0.05,
    color = black );
```

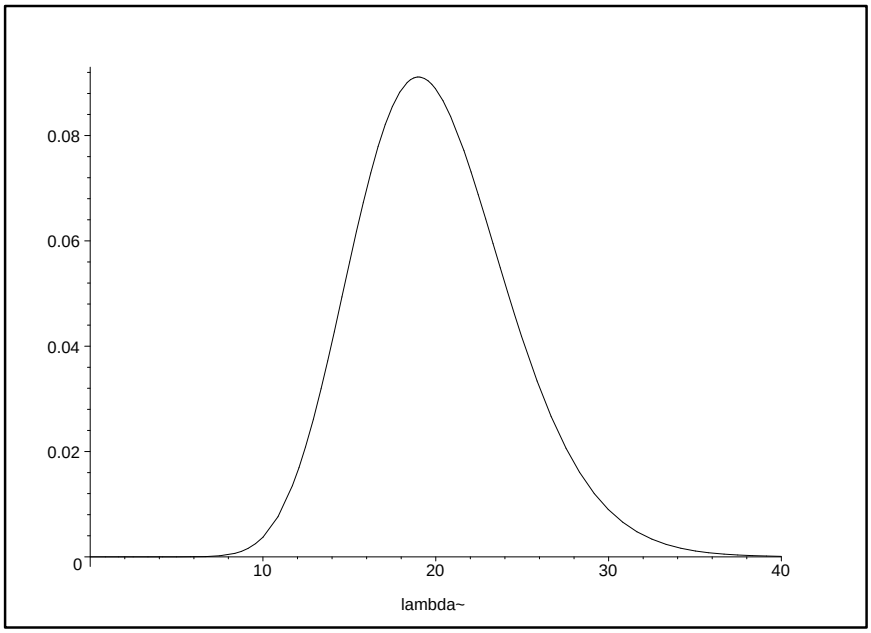



```
> plot( p( lambda, 20, 1 ), lambda = 0 .. 4, color = black );
```



Diffuse Priors (continued)

```
> plot( p( lambda, 20, 1 ), lambda = 0 .. 40, color = black );
```



$\Gamma(20, 1)$ does indeed look **not far from Gaussian**, and at first it may appear that it is indeed **relatively flat** in the region where the likelihood is appreciable ($\lambda \in (1.0, 3.3)$), but we'll see below that it's actually **rather more informative** than we intend.

Recalling that the **mean** and **SD** of a $\Gamma(\alpha, \beta)$ random quantity are $\frac{\alpha}{\beta}$ and $\sqrt{\frac{\alpha}{\beta^2}}$, respectively, and that when used as a prior with the Poisson likelihood the $\Gamma(\alpha, \beta)$ distribution acts like a dataset with **prior sample size** β, you can construct the following table:

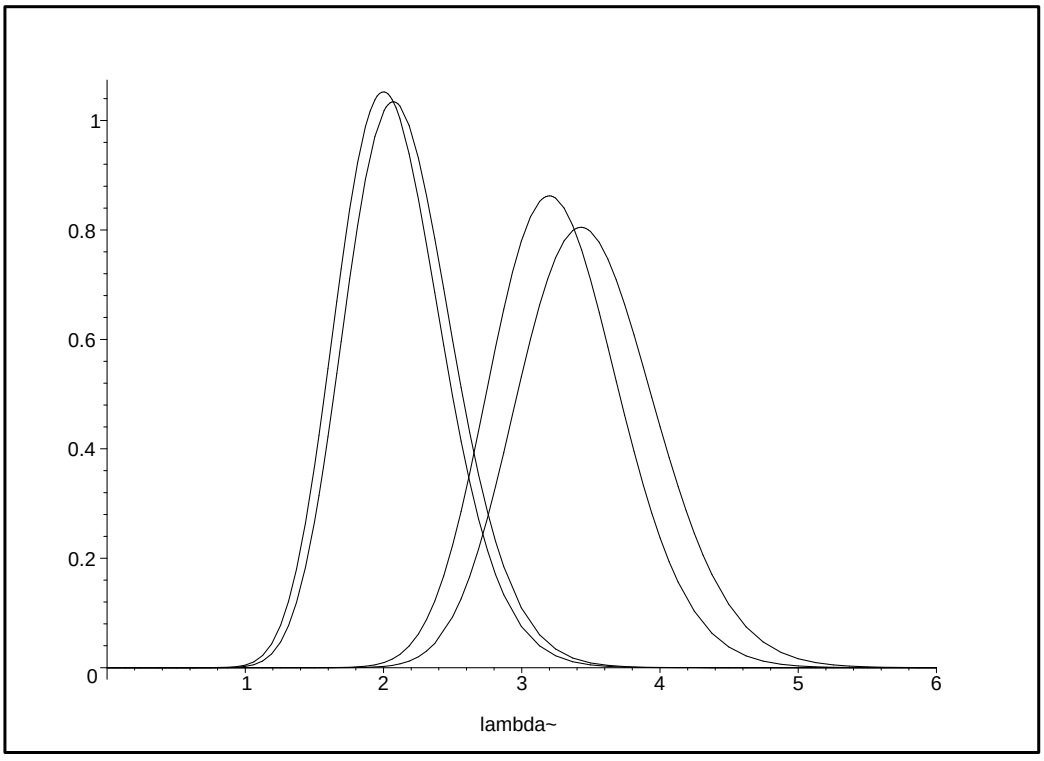
Prior				Posterior			
α	$\beta =$ Sample Size	Mean	SD	α^*	β^*	Mean	SD
0.001	0.001	1	31.6	29.001	14.001	2.071	0.385
1	0.001	1000	1000	30	14.001	2.143	0.391
20	1	20	4.47	49	15	3.267	0.467
20	0.001	20000	4472	49	14.001	3.500	0.500
$U(0, C)$ for $C > 4$		$\frac{C}{2}$	$\frac{C}{\sqrt{12}}$	30	14	2.143	0.391

Diffuse Priors (continued)

The $\Gamma(1, \epsilon)$ prior leads to an analysis that's **essentially equivalent** to the **integrated likelihood (fiducial)** approach back on p. 13, and the $U(0, C)$ prior for $C > 4$ (say) produces similar results: $U(0, C)$ yields the $\Gamma(s + 1, n)$ posterior **truncated** to the right of C (and this truncation has **no effect** if you choose C big enough).

You might say that the $U(0, C)$ distribution has a **prior sample size of 0** in this analysis, and its prior mean $\frac{C}{2}$ and SD $\frac{C}{\sqrt{12}}$ (both of which can be made arbitrarily large by letting C grow without bound) are **irrelevant** (this is an example of how intuition can change when you depart from the class of **conjugate** priors).

```
> plot( { p( lambda, 29.001, 14.001 ), p( lambda, 30, 14.001 ),
    p( lambda, 49, 15 ), p( lambda, 49, 14.001 ) }, lambda = 0 .. 6,
    color = black );
```



The **moral** is that with only $n = 14$ observations, some care is needed (e.g., through **pre-posterior** analysis) to achieve a prior that **doesn't affect the posterior very much**, if that's your goal.

References

Bernardo JM, Smith AFM (1994). *Bayesian Theory*. New York: Wiley.

Craig PS, Goldstein M, Seheult AH, Smith JA (1997). Constructing partial prior specifications for models of complex physical systems. *The Statistician*, **46**, forthcoming.

Draper D (1995). Inference and hierarchical modeling in the social sciences (with discussion). *Journal of Educational and Behavioral Statistics*, **20**, 115–147, 233–239.

Draper D, Hodges JS, Mallows CL, Pregibon D (1993). Exchangeability and data analysis (with discussion). *Journal of the Royal Statistical Society, Series A*, **156**, 9–37.

de Finetti B (1930). Funzione caratteristica di un fenomeno aleatorio. *Mem. Acad. Naz. Lincei*, **4**, 86–133.

de Finetti B (1964). Foresight: its logical laws, its subjective sources. In *Studies in Subjective Probability*, HE Kyburg, Jr., and HE Smokler, eds., New York: Wiley (1980), 93–158.

de Finetti B (1974/5). *Theory of Probability*, **1–2**. New York: Wiley.

Fisher RA (1922). On the mathematical foundations of theoretical statistics. *Philosophical Transactions of the Royal Society of London A*, **222**, 309–368.

Fisher RA (1956). *Statistical Methods and Scientific Inference*. London: Oliver and Boyd.

Freedman D, Pisani R, Purves R, Adhikari A (1998). *Statistics*, third edition. New York: Norton.

Gelman A, Carlin JB, Stern HS, Rubin DB (2003). *Bayesian Data Analysis*, second edition. London: Chapman & Hall.

Hacking I (1975). *The Emergence of Probability*. Cambridge: Cambridge University Press.

Johnson NL, Kotz S (1970). *Distributions in statistics: Continuous univariate distributions*, **1**. New York: Wiley.

Kadane JB, Dickey JM, Winkler RL, Smith WS, Peters SC (1980). Interactive elicitation of opinion for a normal linear model. *Journal of the American Statistical Association*, **75**, 845–854.

References (continued)

Kadane JB, Wolfson LJ (1997). Experiences in elicitation. *The Statistician*, **46**, forthcoming.

Kahn K, Rubenstein L, Draper D, Kosecoff J, Rogers W, Keeler E, Brook R (1990). The effects of the DRG-based Prospective Payment System on quality of care for hospitalized Medicare patients: An introduction to the series. *Journal of the American Medical Association*, **264**, 1953–1955 (with editorial comment, 1995–1997).

Laplace PS (1774). Mémoire sur la probabilité des causes par les évenements. *Mémoires de l'Academie des Sciences de Paris*, **6**, 621–656. English translation in 1986 as "Memoir on the probability of the causes of events," with an introduction by SM Stigler, *Statistical Science*, **1**, 359–378.

O'Hagan A (1997). Eliciting expert beliefs in substantial practical applications. *The Statistician*, **46**, forthcoming.

Samaniego FJ, Reneau DM (1994). Toward a reconciliation of the Bayesian and frequentist approaches to point estimation. *Journal of the American Statistical Association*, **89**, 947–957.

Tierney L, Kadane JB (1986). Accurate approximations for posterior moments and marginal densities. *Journal of the American Statistical Association*, **81**, 82–86.