

Bayesian Model Specification

2: Settings With No Model Uncertainty

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Optimal Sampling-Distribution Specification

You'll recall that **optimal model specification** consists of **conditioning only**, and **exhaustively**, on **propositions rendered true** by the **context** of the **problem** and the **design** of the **data-gathering process**.

In **Day 2 (Lecture Notes, Part 2)** we looked at **optimal prior distribution specification**; what about **sampling distributions**?

Optimal sampling-distribution specification. Sometimes the **sampling distribution** is **uniquely specified** by **problem context**.

These cases are of **two kinds**: based on **theoretical definition-matching** or **exchangeability**.

Case 1: Theoretical Definition-Matching

Example 6. In **random sampling** from a **finite population** with **dichotomous outcomes**, if **You** can **actually achieve** the **theoretical goal** of **sampling at random** either **with** or **without replacement**, then (by **definition**) **You** have **no uncertainty** about the **resulting sampling distribution**: **binomial** with **replacement**, **hypergeometric** without **replacement**.

Example 7. Consider estimating the number $0 < N < \infty$ of individuals in a finite population (such as $\mathcal{P} = \{\text{the deer living on the UCSC campus as of 1 July 2013}\}$).

One popular method for performing this estimation is **capture-recapture sampling**; the simplest version of this approach proceeds as follows.

In **stage I**, a random sample of m_0 individuals is taken, and all of these individuals are tagged and released; then, a short time later, in **stage II** a second independent random sample of n_1 individuals is taken, and the number m_1 of these n_1 individuals who were previously tagged is noted.

If You can actually achieve the theoretical goals of simple random sampling (SRS: at random without replacement) in stage I and IID sampling (at random with replacement) in stage II, then (by definition) the conditional sampling distribution for m_1 given N is

$$(m_1 | N \mathcal{B}) \sim \text{Binomial}\left(n_1, \frac{m_0}{N}\right).$$

Theoretical Definition-Matching (continued)

Example 8. You're watching a counting process unfold in time, looking for the occurrences of specific events; if this process satisfies the following three basic assumptions, then the sampling distribution for the number $N(t)$ of events occurring in $[0, t]$ is (by definition) **Poisson**(λt):

- $P[N(t) = 1|\mathcal{B}] = \lambda t + o(t)$;
- $P[N(t) = 2|\mathcal{B}] = o(t)$;
- The numbers of events in disjoint time intervals are independent.

Example 9. You're watching a counting process unfold in time, keeping track of the elapsed times T_1, T_2, \dots between events; if this process satisfies the three basic assumptions above, then the sampling distribution for the T_i is (by definition) **IID exponential** with mean $\frac{1}{\lambda}$.

Example 9, continued. If the scientific context of the problem ensures that the T_i are **memoryless** — i.e., if $P(T_i > s + t | T_i > t \mathcal{B}) = P(T_i > s | \mathcal{B})$ for all $s, t \geq 0$ — then again

Theoretical Definition-Matching (continued)

(by **definition**) the **sampling distribution** for the T_i is **IID exponential**.

Example 10. Paleobotanists estimate the **moments** in the **remote past** when a **given species first arose** and **then became extinct** by taking **cylindrical, vertical core samples** well **below** the **earth's surface** and **looking for** the **first and last occurrences** of the **species** in the **fossil record**, measured in **meters above** the **unknown point A** at which the **species first emerged**.

Let y_{ij} ($j = 1, \dots, J$) denote the **distance** above A at which **fossil j** is found in **core sample $i \in (1, \dots, I)$** .

Under the scientifically reasonable assumption that **these fossil records** are found at **random points** along the **core sample** (this would be part of B), then **You again** have **no sampling-distribution uncertainty**: by **definition** $(y_{ij} | A B B) \stackrel{\text{IID}}{\sim} \text{Uniform}(A, B)$, where B is the **unknown point** at which the **species went extinct**.

Example 11. The astronomer **John Herschel (1850)** was interested in **characterizing** the **two-dimensional probability distribution** of **errors in measuring** the **position** of a **star**.

Theoretical Definition-Matching (continued)

Let x and y be the **errors** in the **east-west** and **north-south** directions, respectively; Herschel wanted the **joint sampling distribution** $p(x, y|\mathcal{B})$.

He took the following two statements as **axioms**, based on his **astronomical intuition**:

(A_1) **Errors** in **orthogonal directions** should be **independent**, i.e.,
$$p(x, y|\mathcal{B}) = p(x|\mathcal{B}) p(y|\mathcal{B}).$$

An **equivalent expression** for $p(x, y|\mathcal{B})$ is **obtainable** by **transforming** to **polar coordinates**: $p(x, y|\mathcal{B}) = f(r, \theta|\mathcal{B})$.

(A_2) In this **new coordinate system**, the **probability density** of the **errors** should be the **same** no matter **at what angle** the **telescope** is **pointed**; i.e., f should **not depend** on θ , i.e., $f(r, \theta|\mathcal{B}) = f(r|\mathcal{B})$.

He then **showed** that **under** these **two axioms** the **only possible sampling distribution** has x and y as **independently Normal** with **mean 0** and the **same SD** σ .

James Clerk Maxwell (1860) used the **same argument 10 years later**

Sampling Distributions Via Exchangeability

to **characterize** the **unique three-dimensional sampling distribution** of **velocities** of **molecules** in a **gas**.

Case 2: Exchangeability

Example 3 (Day 2, Lecture Notes Part 2, continued). We've already seen an example in which exchangeability led to a **unique sampling distribution**: the **binary mortality indicators** y_i for the **heart attack patients** in **calendar 2014**.

Recall that **de Finetti's Representation Theorem** for **binary outcomes** said **informally** that if **Your uncertainty** about **binary** (y_1, y_2, \dots) is **exchangeable**, then the **only logically-internally-consistent inferential model (prior + sampling distribution)** is

$$\begin{aligned}(\theta|\mathcal{B}) &\sim p(\theta|\mathcal{B}) \\(y_i|\theta\mathcal{B}) &\stackrel{\text{iid}}{\sim} \text{Bernoulli}(\theta),\end{aligned}\tag{1}$$

where θ is **both** the **marginal death probability** $P(y_i = 1|\theta\mathcal{B})$ for **patient** i and the **limiting (population) mean** of (y_1, y_2, \dots) .

Sampling Distributions Via Exchangeability (continued)

This result can be summarized as follows:

For **binary observables** y_i , **exchangeability** + _____ \rightarrow **unique Bernoulli sampling distribution**, where in **this case no additional assumptions** are **needed to fill in the blank**.

This gives rise **immediately** to **questions** like the **following**: what's **needed** in the **blank** to **make this statement true**?

For **non-negative integer observables** y_i ,

$$\text{exchangeability} + \text{_____} \rightarrow \left\{ \begin{array}{l} (\lambda|\mathcal{B}) \sim p(\lambda|\mathcal{B}) \\ (y_i|\lambda \mathcal{B}) \stackrel{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda) \end{array} \right\}. \quad (2)$$

Many people have **worked** on **de-Finetti-style Representation Theorems** of **this type**; here's an **example**.

Example 12. To **get** the **Poisson** result above, the **following assumption** has to **fill in the blank**:

the **conditional distribution** $(y_1, \dots, y_n | s_n \mathcal{B})$, where $s_n = \sum_{i=1}^n y_i$ is a **minimal sufficient statistic** in the **Poisson**(λ) **sampling model**, is **Multinomial** on $\{n\text{-tuples of non-negative integers with sum } s_n\}$ with **Uniform probabilities** $(\frac{1}{n}, \dots, \frac{1}{n})$.

Sampling Distributions Via Exchangeability (continued)

Here are two more examples of this basic idea.

Example 13. For continuous observables y_i on $(0, \infty)$,

$$\text{exchangeability} + \text{_____} \rightarrow \left\{ \begin{array}{l} (\eta|\mathcal{B}) \sim p(\eta|\mathcal{B}) \\ (y_i|\eta\mathcal{B}) \stackrel{\text{IID}}{\sim} \text{Exponential}(\eta) \end{array} \right\}, \quad (3)$$

where _____ is the following:

the conditional distribution $(y_1, \dots, y_n | s_n \mathcal{B})$, where $s_n = \sum_{i=1}^n y_i$ is a minimal sufficient statistic in the **Exponential**(η) sampling model, is **Uniform** on the simplex $\{(y_1, \dots, y_n) : y_i \geq 0 \text{ with } \sum_{i=1}^n y_i = s_n\}$.

Example 14. For continuous observables y_i on $(-\infty, \infty)$,

$$\text{exchangeability} + \text{_____} \rightarrow \left\{ \begin{array}{l} (\sigma|\mathcal{B}) \sim p(\sigma|\mathcal{B}) \\ (y_i|\sigma\mathcal{B}) \stackrel{\text{IID}}{\sim} N(0, \sigma^2) \end{array} \right\}, \quad (4)$$

where _____ is the following:

the conditional distribution $(y_1, \dots, y_n | t_n \mathcal{B})$, where $t_n = \sqrt{\sum_{i=1}^n y_i^2}$ is a minimal sufficient statistic in the $N(0, \sigma^2)$ sampling model,

Sampling Distributions Via Exchangeability (continued)

is **uniform** on the $(n - 1)$ -dimensional sphere of radius t_n in \mathbb{R}^n (this condition is equivalent to the joint distribution $(y_1, \dots, y_n | \mathcal{B})$ being rotationally symmetric).

[short course web page: Singpurwallah (2006), pages 45–57, gives a comprehensive catalog of all known sampling-distribution-via-exchangeability results]

You can see that all of these findings have a **common pattern**:

- (1) You have to be prepared to assume the _____ condition, which is of the form {the conditional distribution of the data vector, given a minimal sufficient statistic in the desired sampling model, is uniform on some space}, and
- (2) You will rarely work on a problem in which that condition is automatically rendered true by the problem context.

This makes the Bernoulli result look like the only useful one arising from exchangeability considerations, but de Finetti (1937) himself proved one more **Representation Theorem** that's even more important and potentially useful than the Bernoulli case:

de Finetti's Representation Theorem for Continuous Outcomes.

You observe (y_1, \dots, y_n) , with the y_i **conceptually continuous** in \mathfrak{R} ;
Your uncertainty about the y_i is **exchangeable**.

If You're prepared to extend Your judgment of exchangeability from (y_1, \dots, y_n) to (y_1, y_2, \dots) , then — **letting** F denote the **empirical cumulative distribution function (CDF)** of the (y_1, y_2, \dots) **values** — the **only logically-internally-consistent inferential model** based on the **observables** is

$$\begin{aligned}(F|\mathcal{B}) &\sim p(F|\mathcal{B}) & (5) \\ (y_i|F\mathcal{B}) &\stackrel{\text{i.i.d.}}{\sim} F.\end{aligned}$$

(**Note that de Finetti's Representation Theorem for binary outcomes** is a **special case** of this result.)

This new theorem requires You to place a scientifically-meaningful prior distribution on the space \mathcal{F} of **all CDFs** on \mathfrak{R} , which **de Finetti didn't have the slightest idea how to do** in **1937**.

Bayesian Qualitative-Quantitative Inference

Putting priors on functions (rather than **scalars, vectors** or **matrices**) is the **subject** addressed by **Bayesian nonparametric methods**; this is an **issue** we'll talk more about in **Part 3** of the **Lecture Notes**.

One more example in which **both** the **prior** and the **sampling distribution** arise **directly** from **problem context**, i.e., in which **optimal Bayesian model specification** is possible:

[short course web page: [Lecture Notes Part 2A \(Bayesian Qualitative-Quantitative Inference\)](#)]