Bayesian Hierarchical Modeling

2: Optimal Prior Distribution Specification

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Short Course (Day 2)
University of Reading (UK)

26 Nov 2014

An Example, to Fix Ideas

Case Study 1. (Krnjajić, Kottas, Draper [KKD] 2008): *In-home geriatric assessment (IHGA)*. In an **experiment** conducted in the **1980s** (Hendriksen et al. 1984), **572 elderly people**, **representative** of $\mathcal{P} =$ {all **non-institutionalized elderly people** in **Denmark**}, were **randomized**, **287** to a **control** (C) group (who received **standard health care**) and **285** to a **treatment** (T) group (who received **standard care plus IHGA**: a kind of **preventive medicine** in which each person's **medical** and **social needs** were assessed and acted upon **individually**).

One **important outcome** was the **number of hospitalizations** during the **two-year** life of the study:

	Number of Hospitalizations						
Group	0	1	...	k	n	Mean	SD
Control	n_{C0}	n_{C1}	...	n_{Ck}	$n_C = 287$	$\bar{y}_C$	s_C
Treatment	n_{T0}	n_{T1}	...	n_{Tk}	$n_T = 285$	$\bar{y}_T$	s_T

Let μ_C and μ_T be the **mean hospitalization rates** (per two years) in $\mathcal{P}$ under the C and T **conditions**, respectively.

Here are **four statistical questions** that **arose** from **this study**:

The Four Principal Statistical Activities

Q_1: Was the **mean number of hospitalizations per two years** in the IHGA group **different from** that in **control** by an **amount** that was **large** in **practical** terms? [**description** involving $\left(\frac{\bar{y}_T - \bar{y}_C}{\bar{y}_C}\right)$]

Q_2: Did IHGA **(causally) change** the **mean number of hospitalizations per two years** by an **amount** that was **large** in **statistical** terms? [**inference** about $\left(\frac{\mu_T - \mu_C}{\mu_C}\right)$]

Q_3: On the **basis** of **this study**, how **accurately** can You **predict** the **total decrease** in **hospitalizations** over a period of N years if **IHGA** were **implemented throughout Denmark**? **[prediction]**

Q_4: On the **basis** of **this study**, is the **decision** to **implement IHGA** throughout Denmark **optimal** from a **cost-benefit** point of view? **[decision-making]**

These questions **encompass** almost all of the **discipline** of **statistics**: **describing** a **data set** D, **generalizing outward inferentially** from D, **predicting new data** D^*, and helping people **make decisions** in the **presence** of **uncertainty** (I include **sampling/experimental design** under **decision-making**; **omitted**: data **quality assurance (QA)**, ...).

An Axiomatization of Statistics

1 **(definition) Statistics** is the study of **uncertainty**: how to **measure it well**, and how to **make good choices** in the face of it.

2 **(definition) Uncertainty** is a state of **incomplete information** about something of interest to **You** (Good, 1950: a **generic person** wishing to **reason sensibly** in the presence of **uncertainty**).

3 **(axiom) (Your uncertainty** about) **"Something of interest to You"** can always be **expressed** in terms of **propositions**: **true/false** statements $A, B, \ldots$

Examples: You may be **uncertain** about the **truth status** of

- $A =$ (**Hillary Clinton** will be **elected U.S. President** in **2016**), or
- $B =$ (the **in-hospital mortality rate** for patients at **hospital** H admitted in **calendar 2010** with a principal diagnosis of **heart attack** was **between 5% and 25%**).

4 **(implication)** It **follows** from 1–3 that **statistics** concerns **Your information** (**NOT** Your **beliefs**) about $A, B, \ldots$

Axiomatization (continued)

5 **(axiom)** But **Your information** cannot be **assessed** in a **vacuum**: all such **assessments** must be made **relative to (conditional on)** Your **background assumptions** and **judgments** about **how the world works** vis à vis $A, B, \ldots$.

6 **(axiom)** These **assumptions** and **judgments**, which are themselves a form of **information**, can always be **expressed** in a **set** $\mathcal{B}$ of **background propositions**, all of which **You believe** to be **true**.

Examples of $\mathcal{B}$:

- In the **IHGA study**, based on the **experimental design**, $\mathcal{B}$ would include the **propositions**

(**Subjects** were **representative of [like a random sample from]** $\mathcal{P}$),

(**Subjects** were **randomized** into one of two groups, **treatment** (**standard care + IHGA**) or **control** (**standard care**)).

7 **(definition)** Call the **"something of interest to You"** θ; in **applications** θ is often a **vector** (or **matrix**, or **array**) of **real numbers**, but **in principle** it could be **almost anything** (a **function**,

Axiomatization (continued)

an **image** of the **surface of Mars**, a **phylogenetic tree**, ...).

IHGA example: θ = **mean relative decrease** $\left(\frac{\mu_T - \mu_C}{\mu_C}\right)$ in **hospitalization rate** in $\mathcal{P}$.

8 **(axiom)** There will typically be an **information source (data set)** D that You judge to be **relevant** to **decreasing** Your uncertainty about θ; in **applications** D is often again a **vector** (or **matrix**, or **array**) of **real numbers**, but **in principle** it too could be **almost anything** (a **movie**, the **words** in a **book**, ...).

9 **(implication)** The **presence** of D creates a **dichotomy**:

- **Your information** about θ {**internal, external**} to D.

(People often talk about a **different dichotomy**: **Your information** about θ {**before, after**} D arrives **(prior, posterior)**, but **temporal considerations** are actually **irrelevant**.)

10 **(implication)** It **follows** from 1–9 that **statistics** concerns itself principally with **five things** (omitted: **description**, **data QA**, ...):

(1) **Quantifying Your information** about θ **internal** to D (given $\mathcal{B}$), and doing so **well** (this term is **not yet defined**);

Foundational Question

(2) **Quantifying Your information** about θ **external** to D (given $\mathcal{B}$), and doing so **well**;

(3) **Combining** these two **information sources** (and doing so **well**) to create a **summary** of **Your uncertainty** about θ (given $\mathcal{B}$) that includes **all available information** You judge to be **relevant** (this is **inference**);

and using **all Your information** about θ (given $\mathcal{B}$) to make

(4) **Predictions** about **future** data values D^* and

(5) **Decisions** about how to **act sensibly**, even though **Your information** about θ may be **incomplete**.

Foundational question: How should these tasks be **accomplished**?

This topic will be **continued** in **Day 3** of this **short course** in **much greater depth**; **for now**, let's **focus** on **task (2)** above — **optimal prior distribution specification** — **first in general**, and **then** with **particular application** to **hierarchical models**.

Optimal Prior Distribution Specification

Sometimes the **prior distribution** is **uniquely specified** by **problem context**; this **most often occurs** when **little or no information** about θ **external** to the **data set** D is **available**.

Example 1. The **simplest situation** is **when** the **unknown** θ **takes on** a **finite number** of **distinct values** $(v_1, \ldots, v_k)$; **these** can **live** in **any space they want**, but **they** can **always** be **mapped** onto k **distinct places** on the **real number line**.

If the **totality** of **Your information** about θ **external** to D is **captured** by the **single proposition**

$$B = \{\theta \text{ takes on a } \textbf{finite} \text{ number of } \textbf{distinct} \text{ values } (v_1, \ldots, v_k)\}, \quad (1)$$

from which it would **follow** that $B = \mathcal{B}$, then **intuition suggests** that the **only possible prior specification** is

$$p(\theta = v_j|\mathcal{B}) = \frac{1}{k} \text{ for all } j = 1, \ldots, k \text{ and 0 otherwise}. \quad (2)$$

Laplace used this **repeatedly** in the **late 1700s without thinking** it **needed any justification**; much **later Keynes** (1921) called it the **Principle of Indifference**.

The Principle of Indifference is Actually a Theorem

However, thinking of **probability** as an **expression** of **Your information** about θ, it's **not just a Principle**, it's a **Theorem**, which (in **light** of its **history**) should be **given** a **new name**; I **propose calling it** the

Uniform Prior Distribution Theorem: **If** the **totality** of **Your information** about θ **external** to D is **captured** by the **single proposition**

$$\mathcal{B} = \{\theta \text{ takes on a } \textbf{finite} \text{ number of } \textbf{distinct} \text{ values } (v_1, \ldots, v_k)\}\,, \quad (3)$$

then Your **only possible logically-internally-consistent prior specification** is

$$p(\theta = v_j|\mathcal{B}) = \frac{1}{k} \text{ for all } j = 1, \ldots, k \text{ and 0 otherwise}\,. \quad (4)$$

One-sentence Proof 1 (by **contradiction**): **Suppose** that $p(\theta = v_j|\mathcal{B})$ is **not constant**; then this **violates** the **assumption** about the **totality** of **Your information**.

More elaborate Proof 2 (by **group invariance**, with a **nod** to **Einstein's relativity proofs**): *[short course web page: Jaynes (2003), pages 37–40]*

Optimal Diffuse Priors

Example 2. The **next simplest situation** is **when** the **unknown** θ **takes on values** in an **interval** on the **real line**, of the **form** (a, b) or $[a, b)$ or $(a, b]$ or $[a, b]$ for some **real numbers** a and b with $-\infty < a < b < \infty$ (**I'll** just **talk** about the **interval** (a, b) from **now on**).

If the **totality** of **Your information** about θ **external** to D is **captured** by the **single proposition**

$$B = \{\{\textbf{Possible values for } \theta\} = \{\text{the } \textbf{interval } (a, b)\}, a < b\}, \quad (5)$$

from which it would **again follow** that $B = \mathcal{B}$, then **intuition again suggests** that the **only possible prior specification** is

$$p(\theta|\mathcal{B}) = \textbf{Uniform}(a, b). \quad (6)$$

This intuition is **based** on the **following "proof" sketch**:

- the **interval** (a, b) can be approximated with $(k + 1)$ **equidistant discrete values** $\left(a, a + \frac{(b-a)}{k}, a + \frac{2(b-a)}{k}, \ldots, b\right)$;
- use the **Uniform Prior Distribution Theorem** above to **conclude** that **all** of **these values** must be **equally likely** (with the **given background information**); and

Optimal Diffuse Priors (continued)

- **pass** to the **limit** as $k \to \infty$.

However, You can **readily see this isn't completely satisfying**, through **simple examples** like the **following**:

Suppose that Your **background knowledge base** $\mathcal{B}$ **about** an **unknown** θ **says only** that θ can **take on all values from 1 to 10**.

This is **equivalent** to the **statement** that $\eta = \frac{1}{\theta}$ can **take on all values from** $\frac{1}{10} = 0.1$ **to 1**.

But a **Uniform prior** on θ **doesn't imply** a **Uniform prior** on η, and **vice versa**, as **You can see** either from the **Change-of-Variables formula** or by **simulation**:

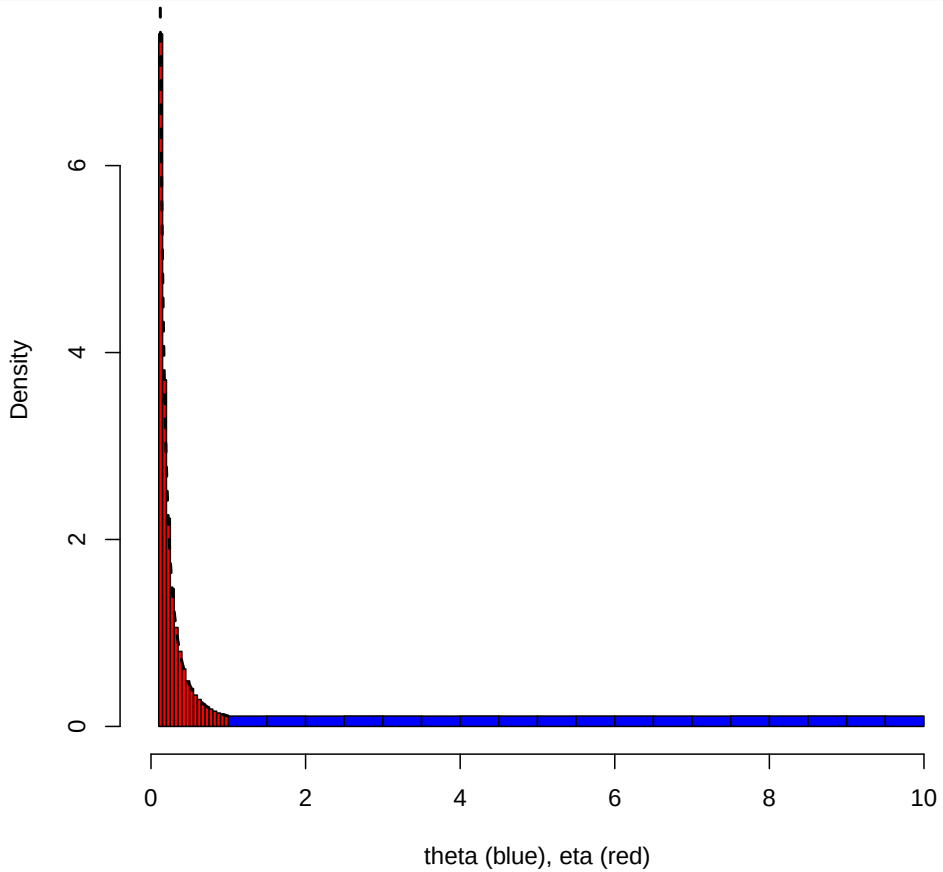
An **easy calculation shows** that $(\theta|\mathcal{B}) \sim$ **Uniform**$(1, 10)$ **iff** $p(\eta|\mathcal{B}) = \frac{1}{9\,\eta^2} \mathsf{I}(0.1, 1)$, and **simulation arrives** at the **same answer**.

The `R` **code** on the **next page stores 1,000,000 simulated draws** from the **Uniform**$(1, 10)$ **distribution** in `theta.star`, **computes** `eta.star` as the **reciprocal** of the `theta.star` **values, plots density-scale histograms** of `theta.star` and `eta.star` on the **same graph**, and **superimposes** the **density** $\frac{1}{9\,\eta^2} \mathsf{I}(0.1, 1)$ on the **histogram** for `eta.star`.

Optimal Diffuse Priors (continued)

```
n <- 1000000
theta.star <- runif( n, 1, 10 )
eta.star <- 1 / theta.star
theta.histogram <- hist( theta.star, plot = F )
eta.histogram <- hist( eta.star, plot = F )
xlim <- c( 0, 10 )
ylim <- range( 0, theta.histogram$intensities,
  eta.histogram$intensities)
# pdf( 'uniform-non-uniform.pdf' )
plot( theta.histogram, xlim = xlim, ylim = ylim,
  col = 'blue', xlab = 'theta (blue), eta (red)', freq = F,
  main = '')
par( new = F )
plot( eta.histogram, xlim = xlim, ylim = ylim,
  xaxt = 'n', yaxt = 'n', col = 'red', add = T,
  freq = F )
eta.grid <- seq( 0.1, 1, length = 500 )
lines( eta.grid, 1 / ( 9 * eta.grid^2 ), lty = 2, lwd = 2 )
# dev.off( )
```

Optimal Diffuse Priors (continued)



Thus, in **settings** in which **all You know** about θ is that it **lives continuously** in (a, b), there is **no unique prior** that **both captures this information** (and **no other information**) and **behaves reasonably** under **arbitrary monotonic transformation** (as **we'll see below**, this **problem bothered Fisher greatly**, and **turned him against** the **Bayesian paradigm**).

Ignorance Priors

Consider the **AMI mortality case study** from the **Day 1 Lecture Notes (Part 2)**, in which **exchangeability** (arising **directly** from **problem context**) led to the **model**

$$\begin{aligned} (\theta|\mathcal{B}) &\sim p(\theta|\mathcal{B}) \\ (y_i|\theta\,\mathcal{B}) &\overset{\text{IID}}{\sim} \text{Bernoulli}(\theta) \end{aligned} \tag{7}$$

for $i = 1, \ldots, n$; **here** y_i is **1** if **AMI patient** i **died within 30 days of admission (0 otherwise)** and θ is the **underlying mortality rate** in the **population** of **patients**, **similar** to **those** in the **study**, to which **You're interested** in **generalizing**.

In this **case study, exchangeability uniquely identifies** the **sampling distribution (Bernoulli)**, but **what about** the **prior** $p(\theta|\mathcal{B})$?

In the **Day 1 Lecture Notes (Part 2)**, we **looked** for a **suitable member** of the **conjugate-prior Beta family**; will **some member** of **this family** be **suitable** as an **ignorance prior**, for **settings** in which **You have little information** about θ **external** to the **data set** y?

Ignorance Priors Via Group Invariance

One possibility — in fact, the **one favored independently** by **both Bayes** and **Laplace** — is of course the **Uniform = Beta**$(1, 1)$ **distribution**, but **we saw above** that **this** is **not invariant** to **monotonic transformation** of θ.

Three other possibilities have been **developed**, each **based** on its own **sensible-sounding principle**.

- **With** the $\mathcal{B}$ of **this problem**, **Jaynes** (2003) *[short course web page: pages 372–386]* uses a **group-invariance argument** to **show** that the **optimal ignorance prior** is

$$p(\theta|\mathcal{B}) \propto \frac{1}{\theta(1-\theta)} = \textbf{Beta}(0, 0)\,. \tag{8}$$

This prior is **improper**, but **leads** to a **proper posterior** for **any data set** in which **at least one 0** and **at least one 1** are **observed**.

However, to **complicate things**, a **different ignorance prior** was **developed** by **Jeffreys** (1939) — with a **different invariance calculation** — to **rebut** an **argument put forward** by **Fisher** (1922) **against** the **Bayes/Laplace prior**:

Fisher Weighs In

"(A) If You're completely ignorant about a success probability θ, aren't You also completely ignorant about any monotone function of θ, such as $\eta = \log \frac{\theta}{1-\theta}$?

"(B) You can't have a Uniform prior on θ and η simultaneously.

"(C) Therefore the entire Bayesian paradigm is rubbish."

Lindley (1954) **qualitative rebuttal**: **if**, on **Your information base**, θ could be **anywhere** in $(0, 1)$, with **no value favored** over **another**, then it's **absurd to say** that **You're equally ignorant** about a **monotone function** such as $\lambda = \theta^{100}$ — **You know for sure** that λ is **close to 0**.

- **Jeffreys** (1939) **quantitative rebuttal**: OK, **Fisher**, You **give me Your sampling distribution** $p(y|\theta\,\mathcal{B})$ (in **any problem** with a **univariate** θ), and **I'll give You** an **ignorance prior** that's **invariant** to **monotone increasing transformation**, namely

$$p(\theta|\mathcal{B}) \propto \sqrt{I(\theta)}\,, \quad \textbf{where} \quad I(\theta) = -E_{(y|\theta\,\mathcal{B})}\left[\frac{\partial^2}{\partial\theta^2}\log p(y|\theta\,\mathcal{B})\right]. \quad (9)$$

Kullback-Leibler Divergence

In the **Bernoulli heart attack mortality example** above, the **Jeffreys idea gives** the **prior**

$$p(\theta|\mathcal{B}) \propto \frac{1}{\sqrt{\theta(1-\theta)}} = \textbf{Beta}(\tfrac{1}{2}, \tfrac{1}{2}) \,, \tag{10}$$

which **differs** from **both** the **Bayes/Laplace prior Beta**$(1, 1)$ and the **Jaynes prior Beta**$(0, 0)$.

- To **make things** even **worse**, **Bernardo** (1979) has **also articulated** a **reasonable-looking principle** that can be **used** to **derive ignorance priors**: **any prior** that's **far away** from the **posterior it leads to** (once the **data** have **arrived**) must have **relatively low information content**.

This **requires** a **notion** of **distance** between **two distributions**.

Definition. The **Kullback-Leibler (KL) divergence** of a **continuous density** $q(\theta|\mathcal{B})$ on $\Re^k$ from **another continuous density** $p(\theta|\mathcal{B})$ on $\Re^k$ — **also known as** the **relative entropy** of q to p — is

$$KL(q||p) = \int_{\Re^k} p(\theta|\mathcal{B}) \log \frac{p(\theta|\mathcal{B})}{q(\theta|\mathcal{B})} \, d\theta \,. \tag{11}$$

Bayes/Laplace, Jaynes, Jeffreys, Bernardo

(**This** is **not a distance metric**, because it's **asymmetric** in q and p, but $\frac{1}{2}\left[KL(q||p) + KL(p||q)\right]$ is a **proper distance metric**.)

Bernardo's idea is to **define** the **reference prior** $p(\theta|\mathcal{B})$ as **that distribution** which **maximizes** the **expectation** (over the **sampling distribution** $p(D|\theta\,\mathcal{B})$) of the **KL divergence** of $p(\theta|\mathcal{B})$ **from** the **posterior distribution** $p(\theta|D\,\mathcal{B})$.

Fact: when $k = 1$, **reference priors** and **Jeffreys priors coincide**, but this is **not necessarily true** for $k > 1$.

So: in the **AMI mortality example, principles** put forth by {**Bayes/Laplace, Jaynes, Jeffreys, Bernardo**} yield the **ignorance priors Beta**$(0, 0)$, **Beta**$(\frac{1}{2}, \frac{1}{2})$ and **Beta**$(1, 1)$.

Remembering that the **prior sample size** in the **Bernoulli sampling model** with the **conjugate** Beta(α, β) prior is $(\alpha + \beta)$, the **prior sample sizes** of the **three ignorance priors** above are **0**, **1** and **2** (**respectively**).

When Principles Collide

Another qualitative rebuttal to Fisher, **based on** those **prior sample sizes** (**many people** have **made this point**, including **Draper** (2009)):

Fisher's point would have **real force if nobody ever collected any data, because** in that case **(posterior = prior)** and **uncertainty** in **how** to **specify** a **diffuse prior can really matter**; but **with even** a **modest amount of data**, the **posteriors** with **prior sample sizes** of **0**, **1** and **2** **will** essentially **coincide**.

When Principles Collide: We **now have three different reasonable-looking principles** for **creating diffuse (ignorance, low-information) priors**: **Jeffreys's Transformation-Invariance**, **Jaynes's Group-Invariance** and **Bernardo's Reference-Prior**.

Q: **What** should You **do** when **reasonable diffuse-prior principles** lead to **priors** that are **different enough** to **matter**?

A: Here's **yet another Principle** that **(greatly) helps**.

The Calibration Principle

Calibration Principle: In **model specification**, it **helps** to **know something** about **how often** {the **methods** You're **using** to choose one **model** over **another**} **get** the **right answer**, and **this** can be **ascertained** by (a) **creating simulation environments** (structurally **similar** to the **setup** of the **scientific problem** You're **currently solving**) in which **You know what the right answer is** and (b) seeing **how often Your methods recover** known **truth**.

The **reasoning** behind the **Calibration Principle** is as follows:

(axiom) You want to **help positively advance** the **course of science**, and **repeatedly getting the wrong answer** runs **counter** to this desire.

(remark) There's **nothing** in the **Bayesian paradigm** to **prevent** You from making **one or both** of the following **mistakes** — (a) choosing $p(D|\theta\,\mathcal{B})$ **badly**; (b) inserting {**strong information** about θ **external** to D} into the **modeling process** that turns out **after the fact** to **have been** (badly) **out of step with reality** — and **repeatedly** doing this **violates the axiom** above.

Reasoning Behind the Calibration Principle

(remark) Paying attention to **calibration** is a **natural activity** from the **frequentist point of view**, but a **desire** to be **well-calibrated** can be **given** an **entirely Bayesian justification** via **decision theory**:

Taking a **broader perspective** over **Your career**, not just within any **single attempt** to solve an **inferential/predictive problem** in collaboration with **other investigators**, Your **desire** to **take part positively** in the **progress of science** can be **quantified** in a **utility function** that **incorporates** a **bonus** for being **well-calibrated**, and in this **context** (Draper, 2013) **calibration-monitoring** emerges as a **natural and inevitable Bayesian activity**.

This **seems to be** a **new idea**: **logical consistency** justifies **Bayesian uncertainty assessment** but **does not provide guidance** on **model specification**; if You accept the **Calibration Principle**, **some** of this **guidance** is **provided**, via **Bayesian decision theory**, through a **desire** on **Your part** to **pay attention to how often You get the right answer**, which is a **central scientific activity**.

A Calibration-Checking Case Study

I **bring up** the **Calibration Principle here** (and **it will come up again** several times **later**) because it **provides** an **answer** to the **question above** about **how** to **resolve conflicts** among **competing diffuse priors**: **You** can **create** a **simulation study**, **just like** the **problem** of **interest to You** but **in which You know** the **parameter values**, and **see which prior does** the **best job** of **correctly recovering known truth**.

Here's an **example** — **Case Study 1 (continued):** *In-home geriatric assessment (IHGA).*

This was the **actual distribution** of **number** of **hospitalizations** over a **two-year period**:

Group	Number of Hospitalizations 0	1	2	3	4	5	6	7	*n*	Mean	SD
Control	138	77	46	12	8	4	0	2	287	0.944	1.24
Treatment	147	83	37	13	3	1	1	0	285	0.768	1.01

(**We'll examine** this **data set** in a **number** of **other ways later**.)

Since the **outcome** in **each group** is a **count** of the **number of occurrences** of a **fairly rare phenomenon**, Your **initial impulse**

Extra-Poisson Variability/Unexplained Heterogeneity

would be to **fit** a **model** in which the **control observations** are **IID Poisson**(λ_C) and the **treatment values** are **independently IID Poisson**(λ_T) (**this** is **called** a **fixed-effects Poisson (FEP) model**).

However, the **Poisson**(λ) **distribution** has **mean** and **variance** both **equal** to λ; in **other words**, for **this distribution** the **variance-to-mean-ratio (VTMR)** is **1**.

Here the **control-** and **treatment-group VTMR values** are $\frac{1.24^2}{0.944} \doteq \mathbf{1.63}$ and $\frac{1.01^2}{0.768} \doteq \mathbf{1.33}$, **respectively**, so the **fixed-effects Poisson model** is **inadequate**.

Count data sets with **VTMR** > 1 are **said** to **exhibit extra-Poisson variability** or **unexplained heterogeneity**.

Consider just the **treatment values** for **now**, and **drop** the T **subscript**.

A **useful way** to **rewrite** the **IID Poisson**(λ_T) **model** for the **observations** $(y_1, \ldots, y_n)$, when **(as is the case here) little** is **known** about **hospitalization rates under IHGA external** to the **data set**, is

Making the Model More Realistic

$$\begin{aligned} (y_i|\lambda_i\,\mathcal{B}) &\overset{\text{indep}}{\sim} \textbf{Poisson}(\lambda_i) \\ \log(\lambda_i) &= \beta_0 \\ (\beta_0|\mathcal{B}) &\sim \textbf{diffuse} \end{aligned} \tag{12}$$

This **moves** toward **scientific realism**, in the **first line** of (12), by **allowing** each **elderly person** to have **her/his own** λ, but this would **create** an **unworkable model** with n **observations** and n **parameters**; the **second line** of the **model (unrealistically) reduces** the **number** of **parameters** from n to 1, by **pretending** that **everybody** in the **treatment group** has the **same underlying rate** λ of **hospitalization**.

In reality it's **far more reasonable** to **think** that **each person** has **his/her own underlying rate** of **hospitalization** that depends on **baseline health status**, **age**, and **various other things**.

If we **had** k **such covariates**, the **second** and **third lines** would be

$$\begin{aligned} \log(\lambda_i) &= \beta_0 + \sum_{j=1}^{k} \beta_j\, x_{ij} \\ (\beta|\mathcal{B}) &\sim \textbf{diffuse} \quad (\beta = (\beta_0, \ldots, \beta_k))\,. \end{aligned}$$

Random-Effects Poisson Model

Now **Hendriksen** (the **study's author**) **forgot to measure** (or **at least** to **report on**) any **covariates**, so the **best we can do** is to **lump** all of these **other latent (unobserved) predictor variables together** into a kind of **"error" term** e_i, as **follows**:

$$\begin{aligned} (y_i|\lambda_i\,\mathcal{B}) &\overset{\text{indep}}{\sim} \textbf{Poisson}(\lambda_i) \\ \log(\lambda_i) &= \beta_0 + e_i \\ (e_i|\sigma\,\mathcal{B}) &\overset{\text{IID}}{\sim} N(0,\sigma^2) \\ (\beta_0\,\sigma|\mathcal{B}) &\sim \textbf{diffuse}\,. \end{aligned} \tag{13}$$

This is **referred to** as a **random-effects Poisson (REP) model**; the **latent variables** e_i are **also called random effects**.

The **Gaussian choice** for the **random-effects distribution** is **conventional**, **not dictated** by the **science** of the **problem** (although **if there were a lot** of **unobserved predictors hidden inside** the e_i, their **weighted sum** would be **close to Gaussian** by the **Central Limit Theorem**).

Model (13) is an **expansion** of the **earlier FEP model** (12), **because** You can **obtain model** (12) from (13) by **setting** $\sigma = 0$,

Unexplained Heterogeneity → Latent Variables

whereas with **(13)** we're **letting** σ **vary** and **learning about it from the data**.

The **addition** of the **random effects** e_i to the **model** is **one way** to **address** the **extra-Poisson variability**: **this model** could **also** be **called** a **lognormal mixture of Poisson distributions**, because **it's as if each person's** λ is **drawn** from a **lognormal distribution** and then **her/his number** of **hospitalizations** y is **drawn** from a **Poisson distribution** with **his/her** λ, and this **hierarchical mixing process** will **make** the **variance** of y **bigger than its mean**.

This is an **example** of a **valuable contemporary modeling approach** — when **unexplained heterogeneity** is **present**, **You** can **use latent variables** to **at least properly quantify it** (of course, **this** is **also** an **old idea**: **that's what** the **"error" term** in **linear regression** is **doing**).

However, a **new challenge** now arises: **how** to **make operational** the **statement** "$(\beta_0\, \sigma|\mathcal{B}) \sim$ **diffuse**".

In **this model** β_0 and σ are **clearly independent given** $\mathcal{B}$ — $p(\beta_0\, \sigma|\mathcal{B}) = p(\beta_0|\mathcal{B})\, p(\sigma|\mathcal{B})$ — so the **new questions** become:

Diffuse Priors on $(-\infty, \infty)$ and $(0, \infty)$

how to **specify** a **diffuse prior** in a **principled way** on (a) $(-\infty, \infty)$ (e.g., a **location parameter** such as β_0) and (b) $(0, \infty)$ (e.g., a **scale parameter** such as σ)?

Example 4. If the **totality** of **Your information** about an **unknown** θ is that **it can take on any value continuously** in $(-\infty, \infty)$, then an **arbitrary shift (left or right)** in the **location** of θ would **leave** the **problem invariant**.

Jaynes (2003) *[short course web page: pages 372–386]* **shows** that the **only prior satisfying** this **invariance property** is **Uniform**$(-\infty, \infty)$.

This is **improper**, but **can be approximated** to **arbitrary accuracy** with a **proper** $N(0, \sigma_\theta^2)$ **prior** with **huge variance** σ_θ^2, or **equivalently tiny precision** $\tau_\theta = \frac{1}{\sigma_\theta^2}$.

(**This** is **also** the **Jeffreys/reference prior** for **this problem**.)

Example 5. If the **totality** of **Your information** about an **unknown** θ is that **it can take on any value continuously** in $(0, \infty)$, then **several cases arise**, and **many of them lead** to **different diffuse priors**

Poisson and Location-Scale Problems

based on **different principles**, e.g.:

(1) **Jaynes** (2003) **shows** that if **You're observing** a **counting process** over **time** that **You're willing** to **model** as **Poisson**, then the **only prior** on the **Poisson rate** $\lambda \in (0, \infty)$ that's **invariant** to **arbitrary rescaling** of **time** by a **constant positive multiple** is $p(\lambda|\mathcal{B}) \propto \frac{1}{\lambda}$, but the **Jeffreys** and **reference priors** for the **IID Poisson model** are **both** $p(\lambda|\mathcal{B}) \propto \frac{1}{\sqrt{\lambda}}$.

(2) As **Jaynes** (2003) **points out**, if the **problem context** (with **real-valued observations** y_i) **leads You** to a **sampling model** of the **form** $(y_i|\mu\,\sigma\,\mathcal{B}) \overset{\text{IID}}{\sim} p(y_i|\mu\,\sigma\,\mathcal{B})$ for **some** p in which μ and σ are **location** and **scale parameters**, **respectively**, and in which **(therefore)** the **problem** should be **invariant** to **arbitrary left-right shifts** and **arbitrary positive rescaling**, the **only prior** that **expresses** this **information** (and **no other external information**) is of the **form** $p(\mu\,\sigma|\mathcal{B}) \propto \frac{1}{\sigma}$ (**this** is **also** the **Jeffreys prior** if $p(\mu\,\sigma|\mathcal{B})$ is **thought of** as $p(\sigma|\mathcal{B})\,p(\mu|\sigma\,\mathcal{B})$ and **Jeffreys's basic idea** is **applied separately** to **each term** in the **product**, and **it's also** the **reference prior** in **this setting**).

Hierarchical Models With Random Effects

However, if **instead** (but **still** with **real-valued observations** y_i) a **unique origin 0** has **special meaning**, but the **problem** is **invariant** to **arbitrary positive rescaling** (e.g., **temperature measured** in **degrees Kelvin**), then the **only prior** that **expresses this information** (and **no other external information**) is of the **form** $p(\mu\,\sigma|\mathcal{B}) \propto \frac{1}{\sigma^2}$.

(3) A **class** of **problems in which** there's **clear uncertainty** about **how** to **arrive** at a **good diffuse prior** is **hierarchical models** with **random effects**, such as the **REP model** above.

$$\begin{aligned} (y_i|\lambda_i\,\mathcal{B}) &\overset{\text{indep}}{\sim} \textbf{Poisson}(\lambda_i) \\ \log(\lambda_i) &= \beta_0 + e_i \\ (e_i|\sigma\,\mathcal{B}) &\overset{\text{IID}}{\sim} N(0,\sigma^2) \\ (\beta_0\,\sigma|\mathcal{B}) &\sim \textbf{diffuse}\,. \end{aligned} \tag{14}$$

You can **factor** the **prior** as $p(\beta_0\,\sigma|\mathcal{B}) = p(\sigma|\mathcal{B})\,p(\beta_0|\sigma\,\mathcal{B})$ and **take** $p(\beta_0|\sigma\,\mathcal{B}) = p(\beta_0|\mathcal{B})$, **since** the **intent** is to **achieve diffuseness** with **both** β_0 and σ.

But what should **You take** for $p(\beta_0|\mathcal{B})$ and $p(\sigma|\mathcal{B})$, if **scientific context** (as it **does here**) implies **diffuseness**?

Diffuse Priors for Hierarchical Models

Simulation studies (Draper, 2013) with **this model** have **shown two things**:

- The **Normal**$(0, \sigma^2_{\beta_0})$ **prior** for β_0 with **small** $\tau_{\beta_0} = \frac{1}{\sigma^2_{\beta_0}}$ produces **good calibration** across a **wide range** of τ_{β_0} **values**, as **long** as τ_{β_0} is **small enough**.

For example, $(\beta_0|\mathcal{B}) \sim N(0, \tau_{\beta_0} = 10^{-6})$ **yields** a **Gaussian prior** with a **mean** of **0** and an **SD** of **1,000**; this **will be** a **flat prior** if the **likelihood** for β_0 is **concentrated** (say) between **−1** and **+1**, but **not** if it's **concentrated** (say) between **−5,000** and **+5,000**.

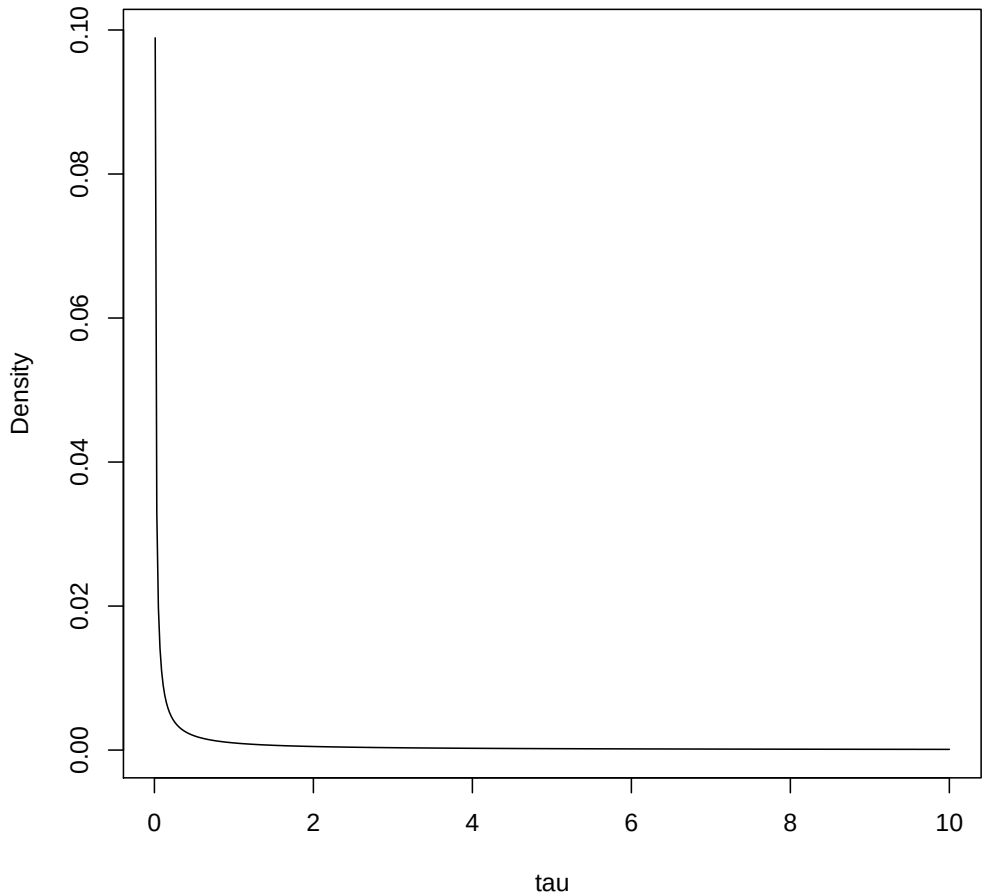
In the **IHGA data set**, **all** of the λ **values**, which are **approximately** e^{β_0}, are **not far from 1**, which **makes** β_0 **close to 0**, so the $N(0, \tau_{\beta_0} = 10^{-6})$ **prior** will be **diffuse**.

- **Much more care** is **required** in **specifying** a **diffuse prior** for σ, or **equivalently** for $\tau = \frac{1}{\sigma^2}$, to **achieve good calibration**.

The **choice** $(\tau|\mathcal{B}) \sim \Gamma(\epsilon, \epsilon)$, for a **small value** of ϵ **such as** 10^{-3}, has been **heavily popularized** by the `WinBUGS` people (**for instance**, it's **still frequently used** in the **examples volumes distributed** with `WinBUGS`).

$\Gamma(\epsilon, \epsilon)$ Prior on τ

This prior has a **mean** for τ of **1** and an **SD** of $\sqrt{10^3} \doteq$ **32**, and **does look flat** over a **broad range** of $\Re$, but (to **stay proper**) it **goes to** ∞ as $\tau \to 0$:

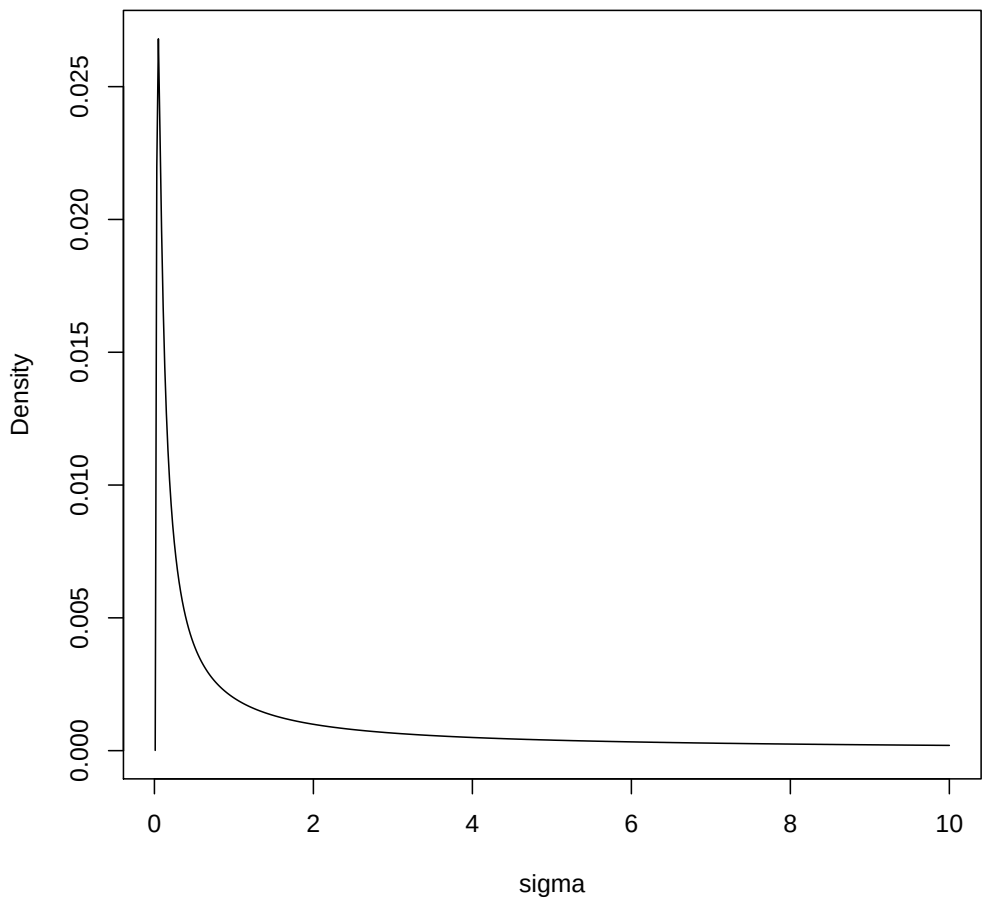


Now the **question** is: **what prior** does this **induce** on $\sigma = \frac{1}{\sqrt{\tau}}$?

The Induced Prior on σ

The **change-of-variables formula** **yields** the following **ugly-looking density expression** for σ:

$$p(\sigma|\mathcal{B}) = \frac{2\,\epsilon^{\epsilon}}{\Gamma(\epsilon)}\sigma^{-(2\epsilon+1)}\exp\left(-\frac{\epsilon}{\sigma^2}\right) . \tag{15}$$



Uniform Prior on σ

Surprisingly, the **tail** of the $\Gamma(\epsilon, \epsilon)$ **prior** on τ is **so heavy** that the **induced prior** on $\sigma = \frac{1}{\sqrt{\tau}}$ **also has** a **spike** near **0**.

From the **plot** on the **previous page**, You can **see what can go wrong** with **this prior**:

if the **likelihood** for σ is **concentrated on small positive values**, then the $\Gamma(\epsilon, \epsilon)$ **prior** on τ will be **unintentionally highly informative**, making the **posterior** on σ **even more concentrated** on **small positive values** than the **likelihood**.

- It **turns out** that a **diffuse prior** that has **good calibration properties** is $(\sigma|\mathcal{B}) \sim$ **Uniform**$(0, C)$, where $C > 0$ is a **constant chosen large enough** to **avoid artificially truncating** the **posterior distribution**.

Ironically, given the **bad results noted above** from the $\Gamma(\epsilon, \epsilon)$ **prior** on τ, it's **still useful** to **employ it** in a **two-step procedure**: You can **determine** a **good value** for C by

(a) **examining** the **preliminary posterior** on σ **produced** by the $\Gamma(\epsilon, \epsilon)$ **prior** on τ, and

Hierarchical Models With Random Effects

(b) **choosing** C — **based on** the **preliminary posterior** in (a) — for a **second** (and **definitive**) **posterior calculation** with the **Uniform**$(0, C)$ **prior** on σ.

[R code on short course web page, illustrating a calibration study comparing these two priors]

Another instance of **calibration sensitivity** to the **form** of the **diffuse prior** occurs with **hierarchical random-effects models** in **meta-analysis**.

Case Study: **Meta-analysis** of **effects** of **low-dose aspirin** on **heart attacks (Lecture Notes, Day 2 Part 1)**.

Recall that the **usual Gaussian random-effects meta-analysis model** in **this study** is

$$\begin{aligned} (\mu\,\sigma|\mathcal{B}) &\sim p(\mu\,\sigma|\mathcal{B}) && \text{(prior)} \\ (\theta_i|\mu\,\sigma\,\mathcal{B}) &\overset{\text{IID}}{\sim} N\left(\mu, \sigma^2\right) && \text{(underlying effects)} \\ (y_i|\theta_i\,\mathcal{B}) &\overset{\text{indep}}{\sim} N(\theta_i, V_i) && \text{(data)}\ . \end{aligned}$$

Well-Calibrated Diffuse Priors in Meta-Analysis

$$
\begin{array}{rcll}
(\mu\,\sigma|\mathcal{B}) & \sim & p(\mu\,\sigma|\mathcal{B}) & \text{(prior)} \\
(\theta_i|\mu\,\sigma\,\mathcal{B}) & \overset{\text{IID}}{\sim} & N\left(\mu,\sigma^2\right) & \text{(underlying effects)} \\
(y_i|\theta_i\,\mathcal{B}) & \overset{\text{indep}}{\sim} & N(\theta_i, V_i) & \text{(data) .}
\end{array}
\tag{16}
$$

Prior specification. The **top level** of **(16)** is **where** the **prior distribution** on the **regression parameters** from the **middle level** is **specified**; **what** should **You choose** for $p(\mu\,\sigma|\mathcal{B})$?

If — as was **true** in **this meta-analysis** — **scientific context** indicates **little information** about (μ, σ) **external** to the **data set, You need** a **good diffuse prior**, where (by the **Calibration Principle**) **"good"** means **well-calibrated**.

[Browne and Draper (2006) on course web page, including Gelman (2006)]

Simulation studies (Gelman 2006, Draper 2013) have **yielded results similar** to **those** with the **REP model above**:

Diffuse Priors in Meta-Analysis (continued)

- The **Normal**$(0, \sigma_\mu^2)$ **prior** for μ with **small** $\tau_\mu = \frac{1}{\sigma_\mu^2}$ produces **good calibration** across a **wide range** of τ_μ **values**, as **long** as τ_μ is **small enough**; here $(\mu|\mathcal{B}) \sim N(0, \tau_\mu = 10^{-6})$ again **works well**.

- **Once** again, **much more care** is **required** in **specifying** a **diffuse prior** for σ, or **equivalently** for $\tau = \frac{1}{\sigma^2}$, to **achieve good calibration**.

The **same Uniform**$(0, C)$ **prior** on σ (with C **chosen well** in the **same way**) **works** here **again**, and Gelman (2006) **has shown** that a **half-**t **prior** (see **his comment** for **details**) also **gives good results**.

[R code on short course web page, illustrating a calibration study comparing a variety of priors]