1 Introduction

The aim of this paper is to describe the class of models which can be implemented using the `R inla` package.

The `R inla` packages solves models using Integrated nested Laplace approximation (INLA) which is a new approach to statistical inference for latent Gaussian Markov random field (GMRF) models described in [Rue et al., 2009].

In short, a latent GMRF model is a hierarchical model where, at the first stage we find a distributional assumption for the observables $\boldsymbol{y}$ usually assumed to be conditionally independent given some latent parameters $\boldsymbol{\eta}$ and, possibly, some additional parameters $\boldsymbol{\theta}_1$

$$\pi(\boldsymbol{y}|\boldsymbol{\eta},\boldsymbol{\theta}_1) = \prod_j \pi(y_j|\eta_j,\boldsymbol{\theta}_1).$$

The third, and last, stage of the model consists of the prior distribution for the hyperparameters $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$. The INLA approach provides a recipe for fast Bayesian inference using accurate approximations to $\pi(\boldsymbol{\theta}|\boldsymbol{y})$ and $\pi(x_i|\boldsymbol{y})$, $i = 0, \ldots, n-1$, i.e. the marginal posterior density for the hyperparameters and the posterior marginal densities for the latent variables. Different types of approximations are available, see [Rue et al., 2009] for details. The approximate posterior marginals can then be used to compute summary statistics of interest, such as posterior means, variances or quantiles.

Using the INLA approach it is also possible to challenge the model itself. The model can be assessed through cross-validation in a reasonable time. Moreover, Bayes factors and deviance information criterion (DIC) can be computed in an efficient way providing tools for model comparison.

2 Model description

The `R inla` library supports hierarchical GMRF models of the following type

$$y_j|\eta_j,\boldsymbol{\theta}_1 \sim \pi(y_j|\eta_j,\boldsymbol{\theta}_1) \qquad j \in J \tag{1}$$

$$\eta_i = \text{Offset}_i + \sum_{k=0}^{n_f-1} w_{ki}\ f_k(c_{ki}) + \boldsymbol{z}_i^T\boldsymbol{\beta} + \epsilon_i \qquad i = 0, \ldots, n_\eta - 1 \tag{2}$$

where

- J is a subset of $\{0, 1, \ldots, n_\eta - 1\}$, that is, not necessarily all latent parameters $\boldsymbol{\eta}$ are observed through the data $\boldsymbol{y}$.
- $\pi(y_j|\eta_j,\boldsymbol{\theta}_1)$ is the likelihood of the observed data assumed to be conditional independent given the latent parameters $\boldsymbol{\eta}$, and, possibly, some additional parameters $\boldsymbol{\theta}_1$. The latent variable η_i enters the likelihood through a known link function.
- $\boldsymbol{\epsilon}$ is a vector of unstructured random effects of length n_η with i.i.d Gaussian priors with precision λ_η:
$$\boldsymbol{\epsilon}|\lambda_\eta \sim \mathcal{N}(\mathbf{0}, \lambda_\eta \boldsymbol{I}) \tag{3}$$
- $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots)$ is a vector of predictors.
- Offset is an a priori known component to be included in the linear predictor during fitting.
- $\boldsymbol{w}_k$ known weights defined for each observed data point.

- $f_k(c_{ki})$ is the effect of a generic covariate k which assumes value c_{ki} for observation i. The functions f_k, $k = 0, \dots, n_f - 1$ comprise usual nonlinear effect of continuous covariates, time trends and seasonal effects, two dimensional surfaces, iid random intercepts and slopes and spatial random effects. The unknown functions, or more exactly the corresponding vector of function evaluations $\boldsymbol{f}_k = (f_{0k}, \dots, f_{(m_k-1)k})^T$, are modelled as GMRFs given some parameters $\boldsymbol{\theta}_{f_k}$, that is
$$\boldsymbol{f}_k|\boldsymbol{\theta}_{f_k} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q}_k^{-1}) \tag{4}$$
- $\boldsymbol{z}_i$ is a vector of n_β covariates assumed to have a linear effect, and is $\boldsymbol{\beta}$ the corresponding vector of unknown parameters with independent zero-mean Gaussian prior with fixed precisions.

The full latent field, of dimension $n = n_\eta + \sum_{j=0}^{n_f-1} m_j + n_\beta$, is then
$$\boldsymbol{x} = (\boldsymbol{\eta}^T, \boldsymbol{f}_0^T, \dots, \boldsymbol{f}_{n_f-1}^T, \boldsymbol{\beta}^T).$$

Note that the latent field $\boldsymbol{x}$ is parametrised using the predictors $\boldsymbol{\eta}$ instead of the unstructured terms $\boldsymbol{\epsilon}$.

All elements of vector $\boldsymbol{x}$ are defined as GMRFs, hence $\boldsymbol{x}$ is itself a GMRF with density:
$$\pi(\boldsymbol{x}|\boldsymbol{\theta}_2) = \prod_{i=0}^{n_\eta-1} \pi(\eta_i|\boldsymbol{f}_0, \dots, \boldsymbol{f}_{n_f-1}, \boldsymbol{\beta}, \lambda_\eta) \prod_{k=0}^{n_f-1} \pi(\boldsymbol{f}_k|\boldsymbol{\kappa}_{f_k}) \prod_{m=0}^{n_\beta-1} \pi(\beta_m) \tag{5}$$
where
$$\eta_i|\boldsymbol{f}_0, \dots, \boldsymbol{f}_{n_f-1}, \boldsymbol{\beta} \sim \mathcal{N}(\sum_{k=0}^{n_f-1} f_k(c_{ki}) + \boldsymbol{z}_i^T\boldsymbol{\beta}, \lambda_\eta) \tag{6}$$
and $\boldsymbol{\theta}_2 = \{\log \lambda_\eta, \boldsymbol{\theta}_{f_0}, \dots, \boldsymbol{\theta}_{n_f-1}\}$ is a vector of unknown hyperparameters.

The last element in the definition of our hierarchical model is a prior distribution for the hyperparameters $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$.

3 Examples

Many well known models from the literature can be written as special cases of (1) and (2)

- *Time series models*

  Time series models are obtained if $c_k = t$ represents time. The functions f_k can model nonlinear trends or seasonal effects
  $$\eta_t = f_{trend}(t) + f_{seasonal}(t) + \boldsymbol{z}_t^T\boldsymbol{\beta}$$
- *Generalised additive models (GAM)*

  A GAM model is obtained if $\pi(y_i|\eta_i, \boldsymbol{\theta}_l)$ belongs to an exponential family, c_k are univariate, continuous covariates and f_k are smooth functions.
- *Generalised additive mixed models (GAMM) for longitudinal data*

  Consider longitudinal data for individuals $i = 0, \dots, n_i - 1$, observed at time points $t_0, t_1, \dots$. A GAMM model extends a GAM by introducing individual specific random effects, i.e.
  $$\eta_{it} = f_0(c_{it0}) + \dots, +f_{n_f-1}(c_{it(n_f-1)})) + b_{0i}w_{it0} + \dots + b_{(n_b-1)i}w_{it(n_b-1)}$$
  where η_{it} is the predictor for individual i at time t, x_{itk}, $k = 0, \dots, n_f - 1$,w_{itq}, $q = 0, \dots, n_b - 1$ are covariate values for individual i at time t, and $b_{0i}, \dots, b_{(n_b-1)i}$ is a vector of n_b individual specific random intercepts (if $w_{itq} = 1$) or slopes. The above model can be written in the general form in equation (2) by defining $r = (i, t)$, $c_{rj} = c_{itj}$ for $j = 0, \dots, n_f - 1$ and $c_{r,(n_f-1)+q} = w_{itq}$, $f_{(n_f-1)+q}(c_{r,(n_f-1)+q}) = b_{qi}w_{itq}$ for $q = 0, \dots, n_b$. In the same way GAMM's for cluster data can be written in the general form (2).

- *Geoadditive models*

  If geographical information for the observations in the data set are available, they might be included in the model as

  $$\eta_i = f_1(c_{0i}) + \cdots + f_{n_f-1}(c_{(n_f-1)i}) + f_{spat}(s_i) + \boldsymbol{z}_i^T \boldsymbol{\beta}$$

  where s_i indicates the location of observation i and f_{spat} is a spatially correlated effect. Models where one of the covariate represent the spatial effect have recently been coined geoadditive by [Kammann and Wand, 2003].

- *ANOVA type interaction model*

  The effect of two continuous covariate w and v can be modelled as

  $$\eta_i = f_1(w_i) + f_2(v_i) + f_{1|2}(w_i, v_i) + \dots$$

  where f_1 and f_2 are the main effects of the two covariates and $f_{1|2}$ is a two dimensional interaction surface. The above model can be written in the general form (2) simply by defining $c_{1i} = w_i$, $c_{2i} = v_i$, $c_{3i} = (w_i, v_i)$,

- *Univariate stochastic volatility model*

  Stochastic volatility models are time series models with Gaussian likelihood where it is the variance, and not the mean of the observed data, to be part of the latent GMRF model. That is

  $$y_i|\eta_i \sim \mathcal{N}(0, \exp(\eta_i))$$

  The latent field is then typically modelled as a autoregressive model of order 1.

References

[Kammann and Wand, 2003] Kammann, E. E. and Wand, M. P. (2003). Geoadditive models. *Journal of the Royal Statistical Society, Series C*, 52(1):1–18.

[Rue et al., 2009] Rue, H., Martino, S., and Chopin, N. (2009). Approximate Bayesian inference for latent Gaussian models using integrated nested Laplace approximations (with discussion). *Journal of the Royal Statistical Society, Series B*, 71(2):319–392.