Likelihood Ratio Tests

For simple vs. simple, N-P says to compare likelihoods.
So perhaps this is a good idea in general?

LRT: For testing $H_0: \theta \in \omega_1$ vs. $H_1: \theta \in \omega_2$
where $\omega_1 \cup \omega_2 = \Omega$

let $$\lambda(\underset{\sim}{x}) = \frac{\sup_{\theta \in \omega_1} L(\theta \mid \underset{\sim}{x})}{\sup_{\theta \in \omega_2} L(\theta \mid \underset{\sim}{x})}$$

The LRT of size α is the test that rejects when $\lambda \le c$, where $\sup_{\theta \in \omega_1} P(\lambda \le c \mid \theta) = \alpha$.

Ex: $X_1, \ldots, X_n \overset{iid}{\sim} U[0, \theta]$.

$H_0: \theta \le 1$ vs. $H_1: \theta > 1$.

Assume $\max\{X_i\} \le 1$ (otherwise, clearly reject).

$$\lambda = \frac{\sup_{\theta \le 1} \frac{1}{\theta^n} I_{\{\max X_i \le \theta\}}}{\sup_{\theta > 1} \frac{1}{\theta^n} I_{\{\max X_i \le \theta\}}} = \frac{\frac{1}{(\max X_i)^n}}{\frac{1}{(1)^n}}$$

LRT is to reject H_0 when $\frac{1}{(\max X_i)^n} \le c$

i.e. when $\max\{X_i\} \ge c'$.

$$\sup_{\theta \le 1} P(\max\{X_i\} \ge c' \mid \theta) = P(\max\{X_i\} \ge c' \mid \theta = 1)$$
$$= 1 - (c')^n = \alpha \qquad c' = (1-\alpha)^{1/n}$$

Ex: $X_1, \ldots, X_n \overset{iid}{\sim} Geom(p)$ $\qquad f_X(x \mid p) = p(1-p)^x \quad \hat{p} = \frac{1}{\bar{x}+1}$

$H_0: p = \frac{1}{2}$
$H_1: p \ne \frac{1}{2}$

$$\lambda = \frac{\sup_{p=\frac{1}{2}} p^n (1-p)^{\sum x_i}}{\sup_{p \ne \frac{1}{2}} p^n (1-p)^{\sum x_i}} = \frac{\left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{\sum x_i}}{\left(\frac{1}{\bar{x}+1}\right)^n \left(1 - \frac{1}{\bar{x}+1}\right)^{\sum x_i}}$$

reject when
$$\lambda = \frac{\left(\frac{1}{2}\right)^{n(\bar{x}+1)}}{\left(\frac{1}{\bar{x}+1}\right)^{n(\bar{x}+1)} (\bar{x})^{n\bar{x}}} < c$$

$$= \frac{\left(\frac{\bar{x}+1}{2}\right)^{n(\bar{x}+1)}}{(\bar{x})^{n\bar{x}}} = \left(\frac{\bar{x}+1}{2\bar{x}}\right)^{n\bar{x}} \left(\frac{\bar{x}+1}{2}\right)^n < c$$

$$\Rightarrow \left(\frac{1}{2} + \frac{1}{2\bar{x}}\right)^{\bar{x}} \left(\frac{\bar{x}+1}{2}\right) < c'$$

$$y^y = e^{y \log y}$$

$$\frac{d y^y}{dy} = e^{y \log y}\left(\frac{y}{y} + \log y\right) = y^y(1 + \log y)$$

$$f(y) = \left(\frac{1}{2} + \frac{1}{2y}\right)^y \left(\frac{y+1}{2}\right) = e^{y \log\left(\frac{1}{2} + \frac{1}{2y}\right)} \left(\frac{y+1}{2}\right)$$

$$f'(y) = \left(\frac{y+1}{2}\right) e^{y \log\left(\frac{1}{2} + \frac{1}{2y}\right)} \left(\log\left(\frac{1}{2} + \frac{1}{2y}\right) + \frac{y}{\frac{1}{2} + \frac{1}{2y}}\left(-\frac{1}{2y^2}\right)\right)$$

$$+ \frac{1}{2} e^{y \log\left(\frac{1}{2} + \frac{1}{2y}\right)}$$

$$= \left(\frac{y+1}{2}\right)\left(\frac{1}{2} + \frac{1}{2y}\right)^y \left(\log\left(\frac{1}{2} + \frac{1}{2y}\right) - \frac{1}{y+1}\right)$$

$$+ \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2y}\right)^y$$

$$= \left(\frac{y+1}{2}\right)\left(\frac{1}{2} + \frac{1}{2y}\right)^y \left(\log\left(\frac{1}{2} + \frac{1}{2y}\right) - \frac{1}{y+1} + \frac{1}{y+1}\right)$$

$$= \left(\frac{y+1}{2}\right)\left(\frac{1}{2} + \frac{1}{2y}\right)^y \log\left(\frac{1}{2} + \frac{1}{2y}\right)$$

~~min~~ max when $f'(y) = 0$, i.e. when $\log\left(\frac{1}{2} + \frac{1}{2y}\right) = 0$

or when $y = 1$

$y = \bar{x} = 1 \Rightarrow \hat{p} = \frac{1}{2}$

$f'(y) < 0$ for $y > 1$

$f'(y) > 0$ for $y < 1$

So reject when $\hat{p}$ is away from $\frac{1}{2}$

reject when $\bar{x}$ is outside some asymmetric interval around 1.

<u>Thm</u>: Under certain regularity conditions (satisfied by exponential families)

$-2 \log \lambda(\underset{\sim}{x}) \xrightarrow{D} \chi^2_p$ $\quad$ $p = \#$ free parameters in H_1 $- \#$ free parameters in H_0