AMS 205 – Homework 2
due in class, Tuesday October 12

1. Let X_1 and X_2 have the joint pdf $f(x_1, x_2) = 15x_1^2 x_2$, $0 < x_1 < x_2 < 1$, zero elsewhere. Find each marginal pdf and compute $\Pr(X_1 + X_2 \leq 1)$. *Hint*: Graph the space of X_1 and X_2 and carefully choose the limits of integration in determining each marginal pdf.

2. Let $f(x_1, x_2) = 21x_1^2 x_2^3$, $0 < x_1 < x_2 < 1$, zero elsewhere, be the joint pdf of X_1 and X_2.
   (a) Find the conditional mean and variance of X_1 given $X_2 = x_2$, $0 < x_2 < 1$.
   (b) Find the distribution of $Y = E[X_1|X_2]$.
   (c) Determine $E[Y]$ and $\text{Var}(Y)$ and compare these to $E[X_1]$ and $\text{Var}(X_1)$ respectively.

3. Let $f(x)$ and $F(x)$ denote, respectively, the pdf and the distribution function of the random variable X. The conditional pdf of X, given $X > x_0$, x_0 a fixed number, is defined by $f(x|X > x_0) = f(x)/[1 - F(x_0)]$, $x_0 < x$, zero elsewhere. This kind of conditional pdf finds application in a problem of time until failure or death, given survival until time x_0.
   (a) Show that $f(x|X > x_0)$ is a pdf.
   (b) Let $f(x) = \exp(-x)$, $0 < x < \infty$, zero elsewhere. Compute $\Pr(X > 2|X > 1)$.

4. Let X and Y have joint pdf $f(x, y) = 1$, $-x < y < x$, $0 < x < 1$, zero elsewhere. Show that, on the set of positive probability density, the graph of $E[Y|x]$ is a straight line, while that of $E[X|y]$ is not a straight line.

5. If $f(x_1, x_2) = \exp(-x_1 - x_2)$, $0 < x_1 < \infty$, $0 < x_2 < \infty$, zero elsewhere, is the joint pdf of the RV's X_1 and X_2, show that X_1 and X_2 are independent and that $M(t_1, t_2) = (1 - t_1)^{-1} * (1 - t_2)^{-1}$, $t_2 < 1, t_1 < 1$. Also show that $E[\exp\{t(X_1 + X_2)\}] = (1 - t)^{-2}$, $t < 1$. Accordingly, find the mean and variance of $Y = X_1 + X_2$.

6. Suppose that a man leaves for work between 8:00 AM and 8:30 AM and takes between 40 and 50 minutes to get to the office. Let X denote the time of departure and let Y denote the time of travel. If we assume that these random variables are independent and uniformly distributed, find the probability that he arrives at the office before 9:00 AM.

7. Let the independent RV's X_1, X_2, and X_3 have the same pdf $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere. Find the probability that exactly two of these three variables exceed 1/2.

8. Let X_1 and X_2 have a trinomial distribution (i.e., there are n independent trials with three possible outcomes with probabilities p_1, p_2, and $1 - p_1 - p_2$, and X_1 is the number of times the first outcome occurred, X_2 the second). Differentiate the moment-generating function to show that their covariance is $-np_1p_2$.

9. Suppose the number of chocolate chips in a certain type of cookies follows a Poisson distribution, and that we want the probability that a cookie of this type contains at least two chips to be at least 0.999. Find the smallest value the mean of the distribution can take.

10. A food manufacturer uses an extruder (a machine that produces bite-size cookies) that yields revenue for the firm at a rate of \$200 per hour when in operation. However, the extruder breaks down an average of two times every day it operates. If Y denotes the number of breakdowns per day, the daily revenue generated by the machine is $R = 1600 - 50 * Y^2$. Find the expected daily revenue for the extruder.