THEOREM

THE AVERAGE HEIGHT a OF A K-ARY TREE HAVING n LEAVES SATISFIES

$$a \geq \log_K(n)$$

COROLLARY

THE AVERAGE HEIGHT a OF A BINARY TREE HAVING n LEAVES SATISFIES

$$a \geq \lg(n).$$

SEE P. 416 OF BRASSARD & BRATLEY FOR PROOF OF COROLLARY $(K=2.)$.

THEOREM.

ANY COMPARISON SORT MUST, IN AVERAGE CASE, AT LEAST $\lg(n!)$ COMPARISONS ON INPUT ARRAYS OF LENGTH n.

COROLLARY

ANY COMPARISON BASED SORTING ALGORITHM RUNS IN (AVERAGE CASE) TIME $\Omega(n \lg n)$.

DECISION TREE ARGUMENTS FOR lower BOUNDS ARE VERY EASY TO USE:

(1) DETERMINE THE MAXIMUM NUMBER OF OUTCOMES TO EACH TEST OF THE INPUT DATA, CALL THAT NUMBER K.

(2) DETERMINE THE NUMBER OF VERDICTS $f(n)$ (i.e. POSSIBLE ALGORITHM OUTPUTS) AS A FUNCTION OF THE INPUT SIZE n.
e.g. SEARCHING : $f(n) = n$
SORTING : $f(n) = n!$

(3) CONCLUDE THAT ANY ALGORITHM WHICH SOLVES THE PROBLEM USING ONLY TESTS OF THE KIND ANALYSED IN (1) MUST DO AT LEAST

WORST CASE : $\lceil \log_k f(n) \rceil$
AVERAGE CASE : $\log_k f(n)$

TESTS ON INPUT OF SIZE n.

<u>NOTE</u>: THESE ARGUMENTS <u>DO NOT</u> ASSERT THE EXISTENCE OF ALGORITHMS WHICH PERFORM THE NUMBER OF TESTS CITED ABOVE. INSTEAD THEY ASSERT THE NON-EXISTENCE OF ALGORITHMS WHICH DO FEWER TESTS.

ADVERSARY ARGUMENTS

CONSIDER AGAIN THE GAME OF 20 QUESTIONS, EXCEPT NOW YOUR OPPONENT CHEATS.

CALL THE PLAYERS A AND B.

A: PRETENDS TO PICK $x \in \{1, \dots, 10^6\}$

B: ASKS A SEQUENCE $Q_1, Q_2, \dots$ OF YES/NO QUESTIONS (EACH QUESTION DEPENDS ON PRECEEDING ANSWERS.)

A: ALWAYS GIVES AN ANSWER WHICH IS CONSISTENT WITH ALL PREVIOUS ANSWERS, BUT WHICH IS DESIGNED TO PROLONG THE GAME AS FAR AS POSSIBLE.

FOR A'S ANSWERS TO BE CONSISTENT, THERE MUST ALWAYS EXIST AN $x \in \{1, \dots, 10^6\}$ SUCH THAT IF x HAD BEEN PICKED (BY AN HONEST PLAYER), THE TRUE ANSWERS WOULD HAVE BEEN THOSE GIVEN BY A.

HOW LONG CAN A KEEP THIS UP? THE ANSWER TO THIS QUESTION PROVIDES A LOWER BOUND ON THE WORST CASE NUMBER OF QUESTIONS WHICH MUST BE ASKED BY ANY ALGORITHM PLAYING THE PART OF B.

WE MUST SPECIFY B'S STRATEGY FOR ANSWERING QUESTIONS.

LET S_i DENOTE THE REMAINING SET OF CANDIDATES FOR THE MYSTERY NUMBER X AFTER THE i^{TH} QUESTION HAS BEEN ASKED (AND ANSWERED.)

e.g. $S_0 = \{1, \ldots, 10^6\}$

LET Q_i = i^{TH} QUESTION, AND LET $A_i(x)$ DENOTE THE (TRUE) ANSWER TO Q_i IF THE MYSTERY NUMBER IS X.

e.g. Q_1 = "is $x \le 500{,}000$"
$A_1(400{,}000)$ = 'YES'
$A_1(600{,}000)$ = 'NO'

LET $Y_i = \{x \in S_{i-1} \mid A_i(x) = \text{'YES'}\}$
$N_i = \{x \in S_{i-1} \mid A_i(x) = \text{'NO'}\}$

e.g. Q_1 = "is $x \le 500{,}000$"
$Y_1 = \{1, \ldots, 500000\}$
$N_1 = \{500001, \ldots, 1000000\}$

NOTE $Y_i \cap N_i = \emptyset$ AND $Y_i \cup N_i = S_{i-1}$. THUS

$$|Y_i| + |N_i| = |S_{i-1}|$$