

LET  $T(n)$  DENOTE THE COST (i.e. TIME) NEEDED TO SORT THE SUB-ARRAY  $A[p \dots r]$ , WHERE  $n = r - p + 1$

THEN  $T(n)$  SATISFIES THE RECURRENCE

$$T(n) = \begin{cases} \Theta(1) & n \leq 1 \\ T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + \Theta(n) & n \geq 2 \end{cases}$$

NOTE THAT THE CONSTANTS HIDDEN BY THE  $\Theta$ -NOTATION AFFECT THE SOLUTION TO THIS RECURRENCE, BUT THEY DO NOT ALTER THE ASYMPTOTIC SOLUTION.

WE WILL SEE IN WHAT FOLLOWS THAT THE ASYMPTOTIC SOLUTION TO THE ABOVE RECURRENCE IS!

$$T(n) = \Theta(n \log n)$$

IN MOST CASES THE ASYMPTOTIC SOLUTION WILL BE SUFFICIENT; OCCASIONALLY WE WILL SEEK AN EXACT SOLUTION.

WE WILL STUDY THREE METHODS:

#### (4.1) SUBSTITUTION METHOD

GUESS AN ASYMPTOTIC (UPPER OR LOWER) BOUND AND TRY TO PROVE IT BY INDUCTION.

#### (4.2) RECURSION TREE / ITERATION METHOD

SUBSTITUTE THE RECURRENCE INTO ITSELF TO OBTAIN A SUMMATION EXPRESSION WHICH CAN BE ANALYSED DIRECTLY, OR CAN BE USED TO OBTAIN A GUESS FOR THE PREVIOUS METHOD.

#### (4.3) MASTER METHOD

APPLY A THEOREM TO FIND ASYMPTOTIC SOLUTIONS TO RECURRENCES OF THE FORM

$$T(n) = \begin{cases} \Theta(1) & n \leq n_0 \\ aT\left(\frac{n}{b}\right) + f(n) & n > n_0 \end{cases}$$

WHERE  $\frac{n}{b}$  REPRESENTS EITHER  $\lfloor \frac{n}{b} \rfloor$  OR  $\lceil \frac{n}{b} \rceil$ .

## 4.1 SUBSTITUTION

$$\text{EX. } T(n) = \begin{cases} 1 & 1 \leq n < 3 \\ 2T(\lfloor n/3 \rfloor) + n & n \geq 3 \end{cases}$$

GUESS !  $T(n) = O(n)$

WE MUST SHOW THAT THERE EXIST POSITIVE NUMBERS  $c$  AND  $n_0$  SUCH THAT :

$$* \quad \forall n \geq n_0 : T(n) \leq cn$$

THE SUBSTITUTION METHOD HAS TWO PARTS

- DETERMINE  $c$  AND  $n_0$  FOR WHICH (\*) CAN BE PROVED BY INDUCTION. WE DO THIS BY MIMICING AN INDUCTIONAL PROOF.
- CARRY OUT A PROOF OF (\*) BY INDUCTION USING VALUES FOR  $c$  AND  $n_0$  FOUND IN THE FIRST PART.

WE OBTAIN  $c=3$ ,  $n_0=1$ . SHOW BY INDUCTION THAT

$$\forall n \geq 1 : T(n) \leq 3n.$$

PROOF:

I.  $T(1) = 1 \leq 3$ , SO THE BASE CASE IS SATISFIED

II. LET  $n > 1$ . ASSUME FOR ALL  $k$  IN THE RANGE  $1 \leq k < n$  THAT  $T(k) \leq 3k$ . THEN

$$\begin{aligned}
T(n) &= 2T(\lfloor n/3 \rfloor) + n \\
&\leq 2 \cdot 3 \lfloor \frac{n}{3} \rfloor + n && \text{(By ind. hyp.)} \\
&\leq 2 \cdot 3 \left( \frac{n}{3} \right) + n \\
&= 2n + n \\
&= 3n.
\end{aligned}$$

$\therefore T(n) \leq 3n$ , AND THE RESULT FOLLOWS FOR ALL  $n$  BY THE INDUCTIVE PRINCIPLE. III.

EXERCISE

SHOW THAT  $T(n) = \Omega(n)$ , WHENCE  $T(n) = \Theta(n)$ .

HINT: USE  $\lfloor x \rfloor > x - 1$  FOR ANY  $x \in \mathbb{R}$ .

