WHEN SHOULD YOU USE DYNAMIC PROGRAMMING?
LOOK FOR THE FOLLOWING:

- OPTIMAL SUBSTRUCTURE (YOU CAN USE DYNAMIC PROGRAMMING.)
- OVERLAPPING SUBPROBLEMS (YOU SHOULD USE DYNAMIC PROGRAMMING.)

MATRIX CHAIN MULTIPLICATION.

CONSIDER THE PROBLEM OF MULTIPLYING TWO MATRICES A $(p \times q)$ AND B $(q \times r)$. THE PRODUCT AB HAS DIMENSIONS $(p \times r)$. IT'S ij ENTRY IS

$$\sum_{k=1}^{q} a_{ik} b_{kj} \qquad (1 \le i \le p,\ 1 \le j \le r)$$

WHICH INVOLVES q SCALAR MULTIPLICATIONS. THUS THE TOTAL NUMBER OF SCALAR MULTIPLICATIONS PERFORMED IN COMPUTING AB IS pqr.

NOW CONSIDER THE PRODUCT OF THREE MATRICES ABC OF DIMENSIONS $(p \times q)$, $(q \times r)$, AND $(r \times s)$ RESPECTIVELY.

THERE ARE TWO POSSIBLE PARENTHESIZATIONS OF THIS PRODUCT :

(1) $A(BC)$ # SCALAR MULTIPLICATIONS $= pqs + qrs$

(2) $(AB)C$ # SCALAR MULTIPLICATIONS $= pqr + prs$

e.g. $p = 10$, $q = 100$, $r = 10$, $s = 100$. THEN (1) YIELDS 200,000 MULTIPLICATIONS AND (2) YIELDS 20,000.

THUS ONE PARENTHESIZATION MAY BE MORE EFFICIENT THAN ANOTHER.

FOR FOUR MATRICES THERE ARE FIVE PARENTHESIZATIONS.

LET $P(n)$ DENOTE THE NUMBER OF DISTINCT PARENTHESIZATIONS OF A PRODUCT OF n MATRICES.

EXCERCISE: SHOW $P(n)$ SATISFIES THE RECURRENCE

$$P(n) = \sum_{k=1}^{n} P(k)P(n-k) \qquad (n \geq 1).$$

THE SEQUENCE $P(n)$ IS CALLED THE CATALAN NUMBERS. ONE CAN SHOW THAT

$$P(n) = \frac{1}{n}\binom{2n-2}{n-1}$$

EXERCISE

USE STIRLING'S FORMULA TO SHOW

$$P(n) = \Theta\left(\frac{4^n}{n^{3/2}}\right)$$

PROBLEM

GIVEN A PRODUCT OF n MATRICES $A_1 A_2 \cdots A_n$ WHERE A_i HAS DIMENSIONS $P_{i-1} \times P_i$, DETERMINE A PARENTHESIZATION WHICH MINIMIZES THE TOTAL NUMBER OF SCALAR MULTIPLICATIONS PERFORMED.

AS USUAL THERE ARE REALLY TWO PROBLEMS

- FIND THE MINIMUM NUMBER OF MULTIPLICATIONS
- FIND THE PARENTHESIZATION WHICH ACHIEVES IT.

WE CAN SEE THAT THE BRUTE FORCE APPROACH IS INEFFICIENT SINCE THE NUMBER OF PARENTHESIZATIONS IS EXPONENTIAL.