8-2-05

<u>THM</u> IF WE MAXIMIZE $\frac{v_i}{w_i}$ in line (6), THEN KNAPSACK RETURNS AN OPTIMAL SOLN.

<u>PROOF</u>:

W.L.O.G. WE ASSUME OBJECTS ARE SORTED BY DECREASING v_i/w_i, i.e.

$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \cdots \geq \frac{v_n}{w_n}$$

LET $x = (x_1, x_2, \ldots, x_n)$ BE THE SOLN RETURNED BY KNAPSACK.

IF ALL $x_i = 1$, THEN clearly x is OPTIMAL. SO ASSUME NOW THAT NOT ALL $x_i = 1$. LET j BE THE FIRST INDEX ST $x_j < 1$. ITS CLEAR THAT

$x_i = 1$ FOR $1 \leq i < j$

$x_j < 1$

$x_i = 0$ FOR $j < i \leq n$

ALSO WE HAVE

$$\sum_{i=1}^{n} x_i w_i = W$$

LET $V(x) = \sum_{i=1}^{n} x_i v_i$. LET $y = (y_1, \ldots, y_n)$ BE ANY OTHER FEASIBLE SOLUTION, AND LET $V(y) = \sum_{i=1}^{n} y_i v_i$. WE MUST SHOW $V(x) \geq V(y)$, WHENCE x IS OPTIMAL.

SINCE y IS FEASIBLE $\sum_{i=1}^{n} y_i w_i \leq W$, SO THAT

$$\sum_{i=1}^{n} (x_i - y_i) w_i = \sum_{i=1}^{n} x_i w_i - \sum_{i=1}^{n} y_i w_i$$

$$= W - \sum_{i=1}^{n} y_i w_i \geq 0.$$

NOW

$$V(x) - V(y) = \sum_{i=1}^{n} x_i v_i - \sum_{i=1}^{n} y_i v_i$$

$$= \sum_{i=1}^{n} (x_i - y_i) v_i$$

$$= \sum_{i=1}^{n} (x_i - y_i) \cdot w_i \cdot \left(\frac{v_i}{w_i}\right)$$

OBSERVE:

- $i < j \Rightarrow x_i = 1 \Rightarrow x_i - y_i \geq 0$ & $\frac{v_i}{w_i} \geq \frac{v_j}{w_j}$

$$\therefore (x_i - y_i) \cdot \left(\frac{v_i}{w_i}\right) \geq (x_i - y_i) \cdot \left(\frac{v_j}{w_j}\right)$$

- $i > j \Rightarrow x_i = 0 \Rightarrow x_i - y_i \leq 0$ & $\frac{v_i}{w_i} \leq \frac{v_j}{w_j}$

$$\therefore (x_i - y_i)\left(\frac{v_i}{w_i}\right) \geq (x_i - y_i)\left(\frac{v_j}{w_j}\right)$$

- $i = j \Rightarrow (x_i - y_i)\left(\frac{v_i}{w_i}\right) = (x_i - y_i)\left(\frac{v_j}{w_j}\right)$

$\therefore$ IN ALL CASES: $\boxed{(x_i - y_i)\left(\frac{v_i}{w_i}\right) \geq (x_i - y_i)\left(\frac{v_j}{w_j}\right)}$

$$\therefore V(x) - V(y) = \sum_{i=1}^{n} (x_i - y_i)\left(\frac{v_i}{w_i}\right) \cdot w_i$$

$$\geq \sum_{i=1}^{n} (x_i - y_i)\left(\frac{v_j}{w_j}\right) w_i$$

$$= \left(\frac{v_j}{w_j}\right) \cdot \sum_{i=1}^{n} (x_i - y_i) w_i \geq 0$$

$\therefore V(x) \geq V(y)$ $\therefore$ x is OPTIMAL.

///.

TRY GREEDY STRATEGY FOR 0-1 KNAPSACK PROB.

<u>EX.</u> $n = 5$, $W = 10$

i	1	2	3	4	5
v_i	2	3	6.6	4	6
w_i	1	2	3	4	5
$\frac{v_i}{w_i}$	2	1.5	2.2	1	1.2

GREEDY SOLN:

$$x = (1 \quad 1 \quad 1 \quad 1 \quad 0) \quad \begin{cases} \text{weight} = 10 \\ \text{Value} = 15.6 \end{cases}$$

<u>EXERCISE</u> SHOW THAT DYN. PROG. YIELDS VERY SAME SOLN.

<u>EX.</u> $n = 5$, $W = 11$

v_i	1	6	18	22	28
w_i	1	2	5	6	7
v_i/w_i	1	3	3.6	3.67	4

$$x = (1 \quad 1 \quad 0 \quad 0 \quad 1) \quad \begin{cases} \text{weight} = 10 \\ \text{VALUE} = 35 \end{cases}$$

EXERCISE: CHECK THAT DYN. PROG. YIELDS SOLN $(0\ 0\ 1\ 1\ 0)$ WITH $\begin{cases} \text{WEIGHT} = 11 \\ \text{VALUE} = 40 \end{cases}$.

COIN CHANGING PROBLEM

GREEDY STRATEGY:

- FROM AMONGST ALL DENOMINATIONS WHOSE ADDITION WOULD NOT CAUSE SUM TO EXCEED N, CHOOSE LARGEST.
- STOP WHEN SUM IS N.

EXERCISE (HARD): SHOW THAT FOR $d = (1, 5, 10, 25, 100)$ THE GREEDY STRATEGY YIELDS AN OPT. SOLN FOR ALL $N \geq 0$.

EXERCISE (EASY). SHOW THAT FOR $d = (1, 10, 25, 100)$ THE GREEDY STRATEGY DOES NOT ALWAYS YIELD AN OPT. SOLN.

EXERCISE (VERY HARD)

CHARACTERIZE ALL DENOMINATION SETS $d = \{d_1, \ldots, d_n\}$ SUCH THAT THE GREEDY STRATEGY ALWAYS YIELDS AN OPTIMAL COIN DISBURSEMENT.

HALLMARKS OF GREEDY APPROACH

- OPTIMAL SUBSTRUCTURE.
- GREEDY CHOICE PROPERTY.

MINIMUM WEIGHT SPANNING TREES

LET $G = (V, E)$ BE AN UNDIR. GRAPH. A GRAPH H IS CALLED A <u>SUBGRAPH</u> OF G IF $V(H) \subseteq V(G)$ AND $E(H) \subseteq E(G)$.

H IS SAID TO BE A SPANNING SUBGRAPH IF ~~&~~ $V(H) = V(G)$. A <u>SPANNING TREE</u> IN G IS A SPANNING SUBGRAPH WHICH IS ALSO A TREE.