Binary tree T.

Proof.

Let T be a binary tree with $h = H(T)$ and $n = L(T)$. We show by induction on h that $h \ge \lceil \lg n \rceil$.

I. If $h = 0$ then T contains just one node, and $n = 0$. Thus in this case $h \ge \lceil \lg n \rceil$.

II. Let $h > 0$ and assume the result holds for any binary tree of height $h-1$. Let T' be the binary tree obtained by deleting all leaves at depth h from T (along with all incident edges.)

Observe that $H(T') = h-1$, and by the induction hypothesis,

$$H(T') \ge \lceil \lg L(T') \rceil .$$

Since each node in T can have at most 2 children, we also have $L(T) \le 2L(T')$, whence

$$L(T') \ge \frac{L(T)}{2} .$$

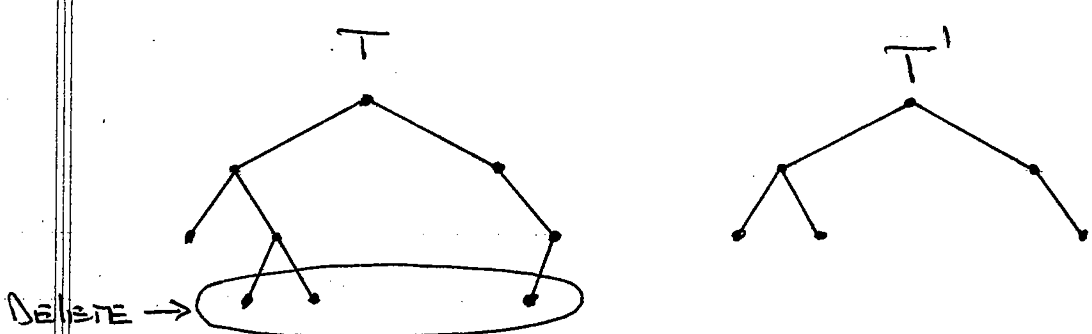
PUTTING THESE INEQUALITIES TOGETHER GIVES

$$\begin{aligned} h-1 &= H(T') \\ &\geq \lceil \lg L(T') \rceil \\ &\geq \left\lceil \lg \frac{L(T)}{2} \right\rceil \\ &= \left\lceil \lg \left(\frac{n}{2}\right) \right\rceil \\ &= \lceil \lg n - 1 \rceil \\ &= \lceil \lg n \rceil - 1 , \end{aligned}$$

AND THEREFORE $h \geq \lceil \lg n \rceil$ AS REQUIRED.

///.

ILLUSTRATION:



$h = H(T) = 3$

$n = L(T) = 4$

$H(T') = 2$

$L(T') = 3$

NOTE ONE CAN DEFINE IN A SIMILAR MANNER COMPLETE AND ALMOST COMPLETE K-ARY TREES, AND PROVE FORMULAS $h = \log_k n$ AND $h = \lceil \log_k n \rceil$ RESPECTIVELY.

<u>EXERCISE</u>: CARRY OUT THE ABOVE DEFINITIONS AND PROOFS.

<u>THEOREM</u>.
THE HEIGHT h OF ANY K-ARY TREE WITH n LEAVES SATISFIES

$$h \geq \lceil \log_k n \rceil$$

<u>EXERCISE</u>: PROVE THIS BY INDUCTION ON THE HEIGHT h.

A <u>DECISION TREE</u> IS A WAY TO REPRESENT THE WORKING OF AN ALGORITHM ON ALL POSSIBLE INPUTS OF A GIVEN SIZE, USING A K-ARY TREE.

EACH INTERNAL NODE REPRESENTS AN OPERATION OR TEST OF SOME KIND ON THE INPUT DATA. EACH LEAF REPRESENTS AN OUTPUT OR <u>VERDICT</u>.

EACH DOWNWARD PATH FROM THE ROOT TO A LEAF REPRESENTS A PARTICULAR SEQUENCE OF TESTS LEADING TO A CONCLUSION, i.e. A PARTICULAR LOGICAL PATHWAY TAKEN BY THE ALGORITHM.

K = MAXIMUM NUMBER OF POSSIBLE OUTCOMES TO EACH TEST

n = NUMBER OF LEAVES
= NUMBER OF POSSIBLE VERDICTS.

h = HEIGHT
= MAXIMUM NUMBER OF TESTS NECESSARY TO REACH A VERDICT.

EX

WE RETURN TO THE EXAMPLE OF FINDING $m \in \{1, \ldots, 6\}$ BY ASKING ONLY YES/NO QUESTIONS. IS THERE AN ALGORITHM WHICH DETERMINES (ANY) m USING AT MOST 2 QUESTIONS?

\# VERDICTS $= n = 6$

\# OUTCOMES ON EACH TEST $= K = 2$

MAX \# OF QUESTIONS $= h$.

BY THE PREVIOUS THEOREM $h \geq \lceil \lg 6 \rceil = 3$.